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A MATHEMATICAL INVESTIGATION OF SOME ECONOMIC EFFECTS OF PROFIT SHARING IN SOCIALIST FIRMS

BY J. KORNAI AND T. LIPTÁK[1]

This article examines some economic problems concerned with profit sharing in a Socialist economy. Two alternative systems of incentives have been made the subject of parallel investigations, comparing the effects which each of these systems have on the firm's behavior. We also examine some problems of price regulation. In the investigation both linear and nonlinear programming methods have been used.

SINCE NATIONALISATION, the working people of Hungarian plants have had a direct financial interest in increasing the firm's profits. So called "directors' funds" have been established for the employees as one of the sources of rewards, and their magnitude depends partly on the size of the profits that are obtained. This financial interest was, however, fairly slight up to 1957. In the Hungarian light industry, for example, the annual sum available for rewards from the directors' funds was, on a per capita basis, the equivalent of 3 to 4 days' earnings per person.[2]

In the years 1957–1958 numerous important changes took place in the administration of Hungarian firms and in the system of financial incentives. One of the most important changes was the introduction of profit sharing. The annual profit share distributed in the light industries on the basis of the results for 1957 corresponded to 17 days' earnings, and on the basis of the results for 1958, to 12 days' earnings.

Profit sharing has proved to be a highly effective incentive. It is, of course, not the main driving force behind the activities of firms. There can be no doubt, however, that decisions taken by firms, or rather by their heads within their own spheres of authority, are considerably influenced by the desire to increase profits.

The introduction of profit sharing into the system of a Socialist planned economy raises a number of theoretical and practical problems. Widespread discussion has occurred, in which both theoretical economists and leading personnel in practical economic administration have participated. For a long time there was an effort to investigate these problems mainly by *empirica* methods. It has, however, seemed apposite also to approach some of th problems of profit-sharing incentives along different paths, in particular

[1] The authors are indebted to Professor A. Rényi, M. Simán, A. Bródy, Dr. J. Pe and L. Geszti of Budapest, Hungary for helpful suggestions. We wish also to thank Professor H. J. A. Kreyberg of Trondheim, Norway for valuable comments on this paper.

[2] See J. Kornai, *Overcentralization in Economic Administration*, Oxford University Press, 1959, p. 101.

analytical methods. This is what has been undertaken in the present investigation, which was conducted in the Hungarian textile weaving mills.[3] As far as the authors know, this is the first attempt in the Hungarian economics literature to clarify the disputed problems of financial incentives by the use of mathematical methods.

1. THE PROBLEM

In this investigation both *linear and nonlinear programming methods* have been used. The mathematical apparatus is itself fairly simple. The reader may, however, consider the *approach* to the problem to be novel.

When an economist in a capitalist country constructs a mathematical model of a programming scheme, he must first establish the true business interests of the economic unit, e.g., the firm being investigated. These interests are then expressed in the objective function of his programming. The business interests of the firm are *given*, determined by the relations of production and economic processes. The task of the economist is mainly to translate the given business interest into the language of mathematics.

The situation in a Socialist economy is different. Here the concrete interests of the firm are shaped by the government, through the measures the government takes. If bonuses paid to the directors and other heads of firms are made by the government to depend mainly on increasing the quantity of production, as was generally the case between 1950 and 1956, then the "firms' interest" is mainly to achieve maximum output. If, on the other hand, the government attaches larger financial rewards to increasing profits, then the "firm's interest" will be mainly to achieve maximum profits, and so forth. The wage and bonus systems and profit sharing are *instruments* in the hands of the State to guide the activities of the firm so as to correspond to the plan for the whole economy.

It hence follows that the problem was not investigated simply from the point of view of the present interests of the firm. Two alternative systems of incentives were made the subject of a parallel investigation, and attention was focused precisely on comparing the effects that each of these systems has on the firm's behaviour. Thus, the purpose was not merely to provide practical advice to the firm on how to achieve an increase in its profit share within the given system of incentives (although it was regarded as important

The authors have, on a commission from the Hungarian Ministry of Light Industries, written a longer study which has appeared as a book under the title "The Mathematical Investigation of Profit Sharing Incentives" (*A nyereségérdekeltség matematikai vizsgálata*), Budapest, 1959, Közgazdasági és Jogi Könyvkiadó. This paper contains some of the findings of that book, considerably abbreviated. For the sake of brevity, the authors have omitted detailed mathematical deductions. Numerous problems discussed in the book are here not mentioned at all.

to develop practical numerical-graphical programming procedures that could be used by firms). The purpose was also to give the economic administration a basis for deciding which incentive system to choose if it wishes the interests of the firm to conform as closely as possible to the general economic policies of the State.

This is related to the broader problem of whether the microeconomic optimum determined by programming for a firm simultaneously ensures the attainment of the macroeconomic optimum, or not. The present investigation did not pose this problem in its entirety. In this respect the scope of research was limited in the following manner:

(1) The system of profit sharing is only one of the instruments of the economic administration. There are also numerous other means available, e.g., direct plan instructions, the central distribution of certain raw materials, the credit system, the central control of prices, etc. Whether the operations of the firm are favourable from the standpoint of the interests of the economy as a whole depends on the *joint* effect of all factors. The present investigation, however, generally accepted the organizational forms of economic administration, the economic-institutional "milieu" in which the activities of firms are carried on, as given—except for some of the principles of profit sharing and price determination under investigation.

(2) It was presumed that the interests of the economy as a whole are expressed by the economic policies of the central administration. A decision by a firm will, therefore, help to achieve the macroeconomic optimum if it exerts an influence in the direction set by State economic policies.

The present research may, therefore, with these reservations, be regarded as a first, simpler, and inevitably one-sided approximation to the broader scope of the "microeconomic versus macroeconomic optimum" problem.

2. THE MODEL

This investigation is specifically confined to the short-run decisions of the firm.

The firm to be discussed manufactures n varieties of articles. Let $x_1, x_2, \ldots, x_n$ be the *individual outputs* of articles $1, 2, \ldots, n$, respectively. The ensemble of $(x_1, x_2, \ldots, x_n)$ is the output *program*. The quantity

$$x = \sum_{i=1}^{n} x_i \tag{1}$$

is the firm's *total output*.

The unit of output chosen should, if possible, be a *technical* parameter, a natural (not monetary) unit of measurement, which adequately reflects the utilization of machinery, independently of the actual articles for whose production the capacity was used. In the textile weaving mills, which constituted the closer subject of the investigation, the pick count proved to be the most suitable unit of measurement.[4]

The assumptions basic to the model are the following:

Assumption 1. The *total production cost* C incurred by the firm depends solely on the output program. Changes in prices, wages, productivity, etc., are neglected, that is, the principle of "ceteris paribus" is applied.

Assumption 2. Costs within the firm are composed of two parts: *individual costs* and *overall costs*. The overall cost function G depends only on the firm's total output:

$$G = G\left(\sum_{i=1}^{n} x_i\right). \tag{2}$$

On the other hand, the individual cost function K is a homogeneous linear function of the individual outputs of the various articles:

$$K = \sum_{i=1}^{n} k_i x_i . \tag{3}$$

The coefficients $k_1, k_2, \ldots, k_n$ in the above are the *individual unit costs* of the articles concerned. The resultant cost function of the firm is thus of the form

$$C(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} k_i x_i + G\left(\sum_{i=1}^{n} x_i\right). \tag{4}$$

This decomposition of costs differs from that usually applied. As far as the authors know it has not previously been used in output programming models.[5] In their opinion, however, this assumption leads to an acceptable approximation to the actual cost relations pertaining in the textile industry. This has been substantiated by statistical investigations and consultations with the textile engineers of the firms concerned.

[4] The traditional engineering practice that has evolved in the textile weaving mills is to measure the performance of looms by the pick count. The weaving mills perform numerous processes (the preparation of the yarn, etc.), but the decisive phase is weaving itself. The technical performance of the loom may therefore adequately represent the utilization, the degree of employment of the whole mill. It is possible technically to determine the maximum number of picks in a given period of time, and the upper limit of output may thus be determined independently of the actual articles the looms are producing.

[5] It is this decomposition of costs that makes it possible—as will be shown—even in the case of a *nonlinear* cost function to program the manufacture of *several* articles by a relatively simple procedure, as opposed to the usual method of marginal analysis which generally assumes the manufacture of *one* article. This, however, makes it necessary to introduce new concepts and relations, corresponding to the nature of individual and overall costs, instead of the usual ones of average cost, marginal cost, etc.

The decomposition of costs into individual and overall costs and the description of the individual cost function as a homogeneous linear function imply the following simplifications:

(a) All the so-called progressive costs of production are included in the overall costs. A part of these is, in fact, essentially independent of what articles the factory produces, and of the individual characteristics of the various articles. (Progressive costs of this type are, e.g., the cost of lighting during the night shift. This is obviously independent of the exact nature of the articles produced by the night shift.) Another part of progressive costs does, if the truth be told, to a certain extent depend on the composition by articles of the firm's output (e.g., the sum paid for overtime depends not only on the total output of the firm but also on the wage requirements of the articles manufactured during overtime). However, the change in overall costs which—if the firm's total output is fixed—can be caused by changes in the composition of the output by particular articles is not very great and has therefore been neglected to facilitate analysis.[6]

(b) In practice the total number of different articles produced by the firm at any particular time is not a matter of indifference, as far as the costs of the firm are concerned. This relation has also been neglected.

Assumption 3. The function $G(x)$ defined for nonnegative values of x is a strictly positive, monotonically increasing, and differentiable function. (The latter statement, of course, involves some simplification, for in reality costs may rise suddenly at certain points.)

The first *differential overall cost function* (derivative of the overall cost function) $G'(x)$ is U-shaped; indeed, the functions $G(x)$ and $G'(x)$ are defined only for an interval $(0, L)$, where the positive number L is the upper limit of the firm's total output. As $x \to L$, $G(x) \to \infty$ and $G'(x) \to \infty$; furthermore, $G'(x)$ decreases monotonically up to a point $x'_{\min}$ within the above interval, and increases monotonically from this point, i.e., it has an absolute minimum, say $G'_{\min}$, in $x'_{\min}$.

It follows from this assumption that the *overall unit cost* function

$$g(x) = \frac{G(x)}{x} \tag{5}$$

is also a U-shaped function, which has an absolute minimum, say $g_{\min}$, at a point $\varkappa'_{\min}$ $(> x)$, where

$$g_{\min} = \min g(x) = g(\varkappa) = G'(\varkappa) \; . \tag{6}$$

[6] The decomposition of costs into individual and overall costs roughly corresponds to the direct and indirect (overhead) costs in bookkeeping. This is, however, only a rough and not a precise correspondence. Here, according to what has been said, all progressive costs have been included in the "overall" costs, including those which are, from the bookkeeper's point of view, considered as direct costs.

The value of $\varkappa$ is defined as the *normal capacity* of production. Within the interval $(0, \varkappa)$, and only there, the relation

$$g(x) \geqslant G'(x) \tag{7}$$

holds.

Statistical investigations, engineers' calculations, and logical considerations all show that amidst the conditions of the Hungarian textile industry it is appropriate in the case of short-run decisions generally to use U-shaped differential cost functions.[7]

It is, moreover, convenient to introduce a so-called *variable overall cost* function, i.e., the function

$$\bar{G}(x) = G(x) - G(0) \, , \tag{8}$$

and the corresponding *variable overall unit cost function,*

$$\bar{g}(x) = \frac{\bar{G}(x)}{x} = \frac{G(x) - G(0)}{x} \, . \tag{9}$$

The value of the latter is equivalent to the tangent of the chord joining the points of abscissae 0 and x on the curve $G(x)$.

Let the locus of the minimum of the variable overall unit cost be $\bar{x}_{\min}$, and the minimum value $\bar{g}_{\min}$. Then

$$\bar{g}_{\min} = \bar{g}(\bar{x}_{\min}) = \min \bar{g}(x) > 0 \, , \tag{10}$$

$$\bar{g}(\bar{x}_{\min}) = G'(\bar{x}_{\min}) \, . \tag{11}$$

Assumption 4. The firm is free to determine its program. Possible instructions from higher authorities on the details of the output program are, therefore, neglected.

Assumption 5. The firm is guided in determining its program solely by the desire to achieve a maximum value of an *index of profitability.* It has already been pointed out, in the introduction, that this is, in fact, not the case, for the activities of the firm are also influenced by numerous other factors. Assumptions 4 and 5 are needed to illuminate fully the effects that are caused specifically by interest in profits.

Assumption 6. The index of profitability appointed for the firm is either (a) *the firm's profit sum* or (b) *the firm's profit ratio.*

[7] In the authors' book all problems were discussed with the parallel use of two types of cost functions. Apart from the case of the type of function here described, that of a linear (overall) cost function was always considered, for under specified circumstances the use of a linear function may also be justified.

(a) The firm's profit sum P is the difference between the *firm's revenue* A and the firm's costs:

$$P = A - C \,, \tag{12}$$

$$A = \sum_{i=1}^{n} a_i x_i \,. \tag{13}$$

Here $a_1, a_2, \ldots, a_n$ are the unit prices of the various articles.

Let the difference between the unit price and the individual unit costs for the various articles be the *price-margin* for the article and denote this by

$$b_i = a_i - k_i \qquad (i = 1,2,\ldots, n) \,. \tag{14}$$

The firm's profit sum may be written as

$$P = P(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} b_i x_i - G\left(\sum_{i=1}^{n} x_i\right) . \tag{15}$$

(b) The firm's profit ratio H is the ratio of the firm's profit sum to the firm's revenue

$$H = P/A \,. \tag{16}$$

Since from (12), $P = A - C$, it follows that

$$H = 1 - Q \tag{17}$$

where

$$Q = Q(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n} k_i x_i + G\left(\sum_{i=1}^{n} x_i\right)}{\sum_{i=1}^{n} a_i x_i} \tag{18}$$

is the *firm's cost ratio.*

In the present discussion the incentive scheme inducing the firm to attain a maximum profit sum will be called one of *sum incentives* while that inducing it to attain a maximum profit ratio will be one of *ratio incentives.*

These are the two types of incentives most frequently mentioned in discussion of the subject, and for this reason it is precisely their comparison that has been made the focus of this investigation. It should be pointed out that in 1959 sum incentives were applied in three branches of light industry (the silk, furniture, and paper industries), while in the remainder—the greater part of the light industries—ratio incentives were used (in a somewhat modified form).

Assumption 7. The demand of the trade fixes an upper limit to the output of each article that may be produced. The firm therefore has certain (positive) limits $M_1, M_2, \ldots, M_n$, which are the *acceptance limits* for the various articles produced. The program is *feasible* if the individual output of no single article exceeds its acceptance limit.

Assumption 8. Unit prices are given for the firm. This corresponds to the general practice, according to which prices are determined by a central price authority. It is true that in some branches of light industry the role of firms in determining prices has lately increased, but this has been ignored in this investigation.

3. PROGRAMMING

The main difficulty in the mathematical programming task is caused by the fact that the nonlinear function $G(x)$ in the objective function is not uniquely specified, but that only certain properties are assumed. Otherwise, this function is regarded as arbitrary. Thus, in programming, we have to start from the graph (or table) of the function $G(x)$ (or possibly from the graph of its derivative).

Two types of incentives involve two types of objective functions. The programming problem is to determine their extremes under given constraints. In both cases one must solve the following more general extreme-value problem (the introduction of the variable z implies a problem of identical structure, arising in the case of both types of incentive systems):

Maximize the linear function

$$c_0 + \sum_{i=1}^{n} c_i z_i \tag{19}$$

of the variables $z_1, z_2, \ldots, z_n$ with the constraints

$$0 \leqslant z_i \leqslant R_i \qquad (i = 1, 2, \ldots, n) \tag{20}$$

and

$$\sum_{i=1}^{n} z_i = z\,, \tag{21}$$

where $R_1, R_2, \ldots, R_n$ and z ($\leqslant R_1 + R_2 + \ldots + R_n$) are positive constants.

This problem may be easily solved. Let the coefficients $c_1, c_2, \ldots, c_n$ be arranged in order of decreasing magnitude and let $(i_1, i_2, \ldots, i_n)$ be the permutation of the indices $(1, 2, \ldots, n)$ obtained so that

$$c_{i_1} \geqslant c_{i_2} \geqslant \ldots \geqslant c_{i_n}\,. \tag{22}$$

Next, the variables are successively "filled up" in the order of the permutation $(i_1, i_2, \ldots, i_n)$ to their maximum value R_i until their sum exceeds z. The last variable of the sequence that has to be used is given the value remaining from z and the rest are chosen to be 0. In other words, if

$$R_{i_1} + \ldots + R_{i_{m-1}} < z \leqslant R_{i_1} + \ldots + R_{i_{m-1}} + R_{i_m} \tag{23}$$

then the coordinates of the maximum point are

$$z^*_{i_1} = R_{i_1}, \ldots, z^*_{i_{m-1}} = R_{i_{m-1}}\,, \tag{24}$$

$$z^*_{i_m} = z - (R_{i_1} + \ldots + R_{i_{m-1}})\,, \tag{25}$$

$$z^*_{i_{m+1}} = z^*_{i_{m+2}} = \ldots = z^*_{i_n} = 0\,. \tag{26}$$

The above system of $(z^*_1, z^*_2, \ldots, z^*_{n_i})$ is henceforth called "the system filled up with z according to the permutation $(i_1, i_2, \ldots, i_n)$."

It should be noted that if the *minimum* of the function (19), under the constraints (20) and (21), is sought, the solution can be obtained by a process analogous to the above. Let the coefficients $c_1, c_2, \ldots, c_n$ then be arranged in order of *increasing* magnitude and let $(j_1, j_2, \ldots, j_n)$ be the permutations of the ordinals obtained, so that

$$c_{j_1} \leqslant c_{j_2} \leqslant \ldots \leqslant c_{j_n}\,, \qquad (j_1 = i_n, \ldots, j_n = i_1)\,. \tag{27}$$

The minimum point is given by the system filled up with z according to the permutation $(j_1, j_2, \ldots, j_n)$.

(a) *In the case of sum incentives* the programming problem is equivalent to that of finding the optimum program $(x^*_1, \ldots, x^*_n)$ maximizing the expression

$$P(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} b_i x_i - G\Big(\sum_{i=1}^{n} x_i\Big) \tag{28}$$

under the constraints

$$0 \leqslant x_i \leqslant M_i \qquad (i = 1, 2, \ldots, n)\,. \tag{29}$$

Let the supplementary constraint

$$\sum_{i=1}^{n} x_i = x \tag{30}$$

be introduced in addition to (29), where x is provisionally fixed. Since the second term of the right side of (28) is then constant, the profit sum P will, under the constraints (29), (30), attain its maximum at the same place as the linear function

$$B(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} b_i x_i\,. \tag{31}$$

This reduces the problem to the maximization problem solved previously: the solution is obtained by arranging the coefficients b_i in the order of decreasing magnitude

$$b_{i_1} \geqslant b_{i_2} \geqslant \ldots \geqslant b_{i_n} \tag{32}$$

and filling up the program $(x_1, x_2, \ldots, x_n)$ with x according to the above permutation $(i_1, i_2, \ldots, i_n)$. The corresponding value of the linear function (31) is

$$B^*(x) = \max_{\substack{0 \leq x_i \leq M_i \\ \sum_{i=1}^{n} x_i = x}} B(x_1, x_2, \ldots, x_n) \tag{33}$$

$$= \begin{cases} b_{i_1} x \text{ if } 0 \leqslant x \leqslant M_{i_1}, \\ b_{i_1} M_{i_1} + b_{i_2}(x - M_{i_1}) \text{ if } M_{i_1} \leqslant x \leqslant M_{i_1} + M_{i_2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum_{k=1}^{n-1} b_{i_k} M_{i_k} + b_{i_n}(x - \sum_{k=1}^{n-1} M_{i_k}) \text{ if } \sum_{k=1}^{n-1} M_{i_k} \leqslant x \leqslant \sum_{k=1}^{n} M_{i_k}. \end{cases}$$

$B^*(x)$, as a function of x, is concave and linear in the intervals $(\Sigma_{k=1}^{r} M_{i_k}, \Sigma_{k=1}^{r+1} M_{i_k})$, $r = 1, 2, \ldots, n-1$.

The optimum program is obtained by determining the value x^* of x for which the function

$$P^*(x) = \max_{\substack{0 \leq x_i \leq M_i \\ \sum_{i=1}^{n} x_i = x}} P(x_1, x_2, \ldots, x_n) = B^*(x) - G(x) \tag{34}$$

has a maximum, and by filling up the program $(x_1, x_2, \ldots, x_n)$ with x^* according to the permutation $(i_1, i_2, \ldots, i_n)$ defined by (32).

x^* may be determined either by direct reading after plotting the graphs of $B^*(x)$ and $G(x)$, or, analytically, after plotting the corresponding differential curves and obtaining their points of intersection.

Let us first examine the conditions under which there can be a maximum *within* the interval $(0, M_1 + M_2 + \ldots + M_n)$. Applying the usual analytical criterion, the necessary condition that $P^*(x)$ should possess an extreme value at an interior point x^* of the interval is that

$$P^{*\prime}(x^*) = 0, \tag{35}$$

i.e., that

$$G'(x^*) = B^{*\prime}(x^*) \tag{36}$$

should hold. Since $B^{*\prime}(x)$ has a break at the inflection points of the function

$B^*(x)$ which is linear in intervals, condition (36) is taken to mean that the *graph* of the decreasing step-function $B^{*\prime}(x)$ should intersect the graph of $G'(x)$ at the point $x = x^*$ (Figures 1–4).

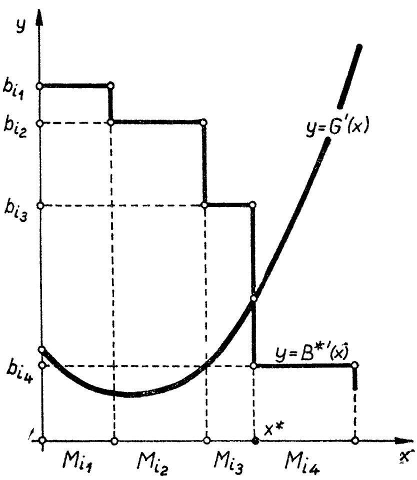

FIGURE 1

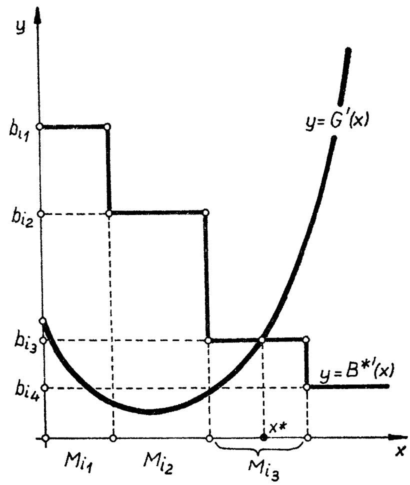

FIGURE 3

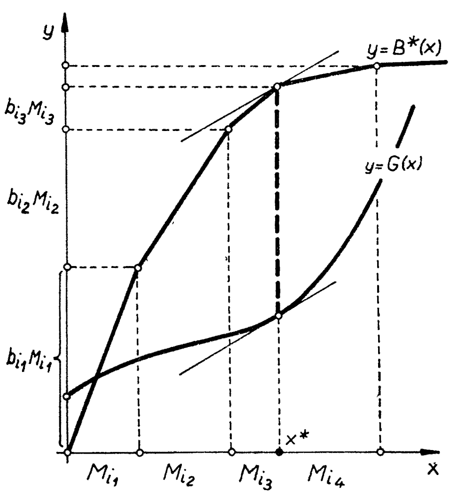

FIGURE 2

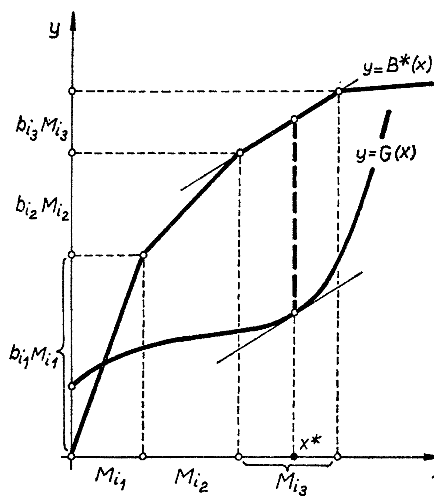

FIGURE 4

A further condition for the maximum is that x^* should also be a point of decrease of $P^{*\prime}(x)$, i.e., that the graph of $B^{*\prime}(x)$ should intersect the graph of $G'(x)$ coming from the plane area above the graph of $G'(x)$ to the plane area below it.

The absolute maximum of $P^*(x)$ is either at one of these points of intersection, or at the beginning or end point. The last case may be eliminated if $b_{i_n} < G'(M_1 + M_2 + \ldots + M_n)$ and the previous case if $b_{i_1} > G'(0)$. The absolute maximum—i.e., the optimal output of the firm—may be determined by comparing the values of $P^*(x)$ at these points.

These latter calculations may be avoided if the equation $G'(x) = B^{*\prime}(x)$ has a solution greater than $\bar{x}_{\min}$ (see (10)–(11) for definition), for in this case the absolute maximum of $P^*(x)$ is certain to be at the point x^*. To see this, let us construct a tangent to the curve of $G(x)$ at the point $x = x^*$. Then, since $x^* > \bar{x}_{\min}$, the curve will proceed throughout *above* the tangent, so that

$$G(x) \geqslant G(x^*) + G'(x^*)(x - x^*)\,. \tag{37}$$

(Equality can occur only if $x = x^*$.) At the same time, because of (36), a tangent (or a supporting line) parallel to the previous tangent may be drawn to the graph $B^*(x)$ at the point $x = x^*$ (Figure 5). Since, furthermore, $B^*(x)$

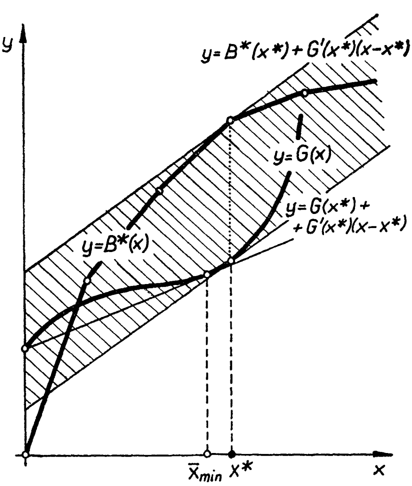

FIGURE 5

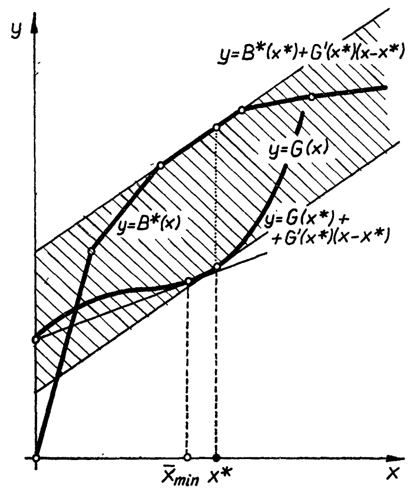

FIGURE 6

is a concave function, its graph proceeds everywhere under this tangent, so that

$$B^*(x) \leqslant B^*(x^*) + G'(x^*)(x - x^*)\,. \tag{38}$$

If the appropriate sides of the inequalities (37) and (38) are substracted one from another, the inequality

$$B^*(x) - G(x) \leqslant B^*(x^*) - G(x^*) \tag{39}$$

is obtained for the entire interval $(0, M_1 + M_2 + \ldots + M_n)$ (equality may only occur at the point $x = x^*$). This actually states that x^* is the locus of the absolute maximum of the function $\mathrm{P}^*(x) \doteq B^*(x) - G(x)$ (Figures 5–6).

(b) *In the case of ratio incentives,* programming involves solving the following mathematical problem: Find the optimum program $(x_1^*, x_2^*, \ldots, x_n^*)$ that makes the expression

$$Q(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n} k_i x_i + G(\sum_{i=1}^{n} x_i)}{\sum_{i=1}^{n} a_i x_i} \tag{40}$$

minimal under the constraints (29).

If $G(x)$ is a linear function this problem can again be reduced to the extreme-value problem previously discussed. For if

$$G(x) = G(0) + tx\,, \tag{41}$$

where $G(0)$ and t are constants, then (40) takes the form

$$Q(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n} (k_i + t) x_i + G(0)}{\sum_{i=1}^{n} a_i x_i}\,. \tag{42}$$

Introducing the new variables

$$y_i = a_i x_i \qquad (i = 1, 2, \ldots, n) \tag{43}$$

and the symbols

$$d_i = \frac{k_i + t}{a_i} \qquad (i = 1, 2, \ldots, n) \tag{44}$$

to represent the *differential cost ratios* $(k_i + t)/a_i$, the problem may be stated as follows: Find the $(y_1^*, y_2^*, \ldots, y_n^*)$ that make the expression

$$R(y_1, y_2, \ldots, y_n) = \frac{\sum_{i=1}^{n} d_i y_i + G(0)}{\sum_{i=1}^{n} y_i} \tag{45}$$

minimal under the constraints

$$0 \leqslant y_i \leqslant a_i M_i \qquad (i = 1, 2, \ldots, n)\,. \tag{46}$$

Introducing the auxiliary constraint

$$\sum_{i=1}^{n} y_i = y \tag{47}$$

(y fixed), the problem can be reduced to that of finding the minimum of the linear function

$$C(y_1, y_2, \ldots, y_n) = G(0) + \sum_{i=1}^{n} d_i y_i \tag{48}$$

under the constraints

$$0 \leqslant y_i \leqslant a_i M_i \qquad (i = 1, 2, \ldots, n)\,, \tag{49}$$

$$\sum_{i=1}^{n} y_i = y\,. \tag{50}$$

This, as has been shown, is the system filled up with y according to the permutation $(j_1, j_2, \ldots, j_n)$ defined by the order

$$d_{j_1} \leqslant d_{j_2} \leqslant \ldots \leqslant d_{j_n} . \tag{51}$$

The corresponding value of the cost function C, from (48), is

$$C^*(y) = G(0) + \begin{cases} d_{j_1} y \text{ if } 0 \leqslant y \leqslant a_{j_1} M_{j_1} , \\ d_{j_1} a_{j_1} M_{j_1} + d_{j_2}(y - a_{j_1} M_{j_1}) \text{ if } a_{j_1} M_{j_1} \leqslant y \leqslant a_{j_1} M_{j_1} + a_{j_2} M_{j_2} , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum_{k=1}^{n-1} d_{j_k} a_{j_k} M_{j_k} + d_{j_n}(y - \sum_{k=1}^{n-1} a_{j_k} M_{j_k}) \text{ if } \sum_{k=1}^{n-1} a_{j_k} M_{j_k} \leqslant y \leqslant \sum_{k=1}^{n} a_{j_k} M_{j_k} . \end{cases} \tag{52}$$

$C^*(y)$ as a function of y is a convex function and linear in the intervals $(\Sigma_{k=1}^{r} a_{jk} M_{jk}, \Sigma_{k=1}^{r+1} a_{jk} M_{jk})$, $r = 0, 1, \ldots, n-1$.

The optimal program is obtained by determining the value y^* of y for which the function

$$R^*(y) = \frac{C^*(y)}{y} \tag{53}$$

has a minimum, by filling up the system $(y_1, y_2, \ldots, y_n)$ with y^* according to the permutation $(j_1, j_2, \ldots, j_n)$ defined in (51), and by computing the optimum program $(x_1^*, x_2^*, \ldots, x_n^*)$ from $(y_1^*, y_2^*, \ldots, y_n^*)$ by the relations

$$x_i^* = \frac{y_i^*}{a_i} \qquad (i = 1, 2, \ldots, n) . \tag{54}$$

Graphically, y^* is the abscissa of the point of tangency of the tangent line (or the supporting line) drawn to the convex function $C^*(y)$ from the origin (Figure 7). The figure shows that possibly any point of an interval may be chosen as y^*. In this case, of course, several optimum programs may exist.

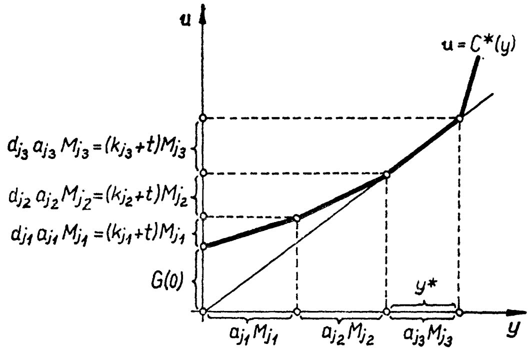

FIGURE 7

If $G'(x)$ is a U-shaped function the problem of programming may be reduced to the linear case, for a simple calculation shows that if $(\overset{*}{x}_1, \overset{*}{x}_2, \ldots, \overset{*}{x}_n)$ is the optimum program and $\overset{*}{x} = \overset{*}{x}_1 + \overset{*}{x}_2 + \ldots + \overset{*}{x}_n$ and $\overset{*}{Q} = Q(\overset{*}{x}_1, \overset{*}{x}_2, \ldots, \overset{*}{x}_n)$, then for the components $\overset{*}{x}_r$ for which

$$\frac{k_r + G'(x^*)}{a_r} < Q^* , \tag{55}$$

$\overset{*}{x}_r = M_r$ necessarily, while for the components $\overset{*}{x}_r$, for which

$$\frac{k_r + G'(x^*)}{a_r} > Q^* , \tag{56}$$

$\overset{*}{x}_r = 0$ must hold. To prove (55), let us take an index r for which $\overset{*}{x}_r < M_r$. If Δ is any number between 0 and $M_r - \overset{*}{x}_r$, then the program $(\overset{*}{x}_1, \ldots, \overset{*}{x}_r - \Delta, \ldots, \overset{*}{x}_n)$ is also feasible, so that the optimality of $(\overset{*}{x}_1, \ldots, \overset{*}{x}_r, \ldots, \overset{*}{x}_n)$ leads to

$$Q(x_1^*, \ldots, x_r^* - \Delta, \ldots, x_n^*) \leqslant Q(x_1^*, \ldots, x_r^*, \ldots, x_n^*) = Q^* . \tag{57}$$

It follows from (40) that

$$Q(x_1^*, \ldots, x_r^* - \Delta, \ldots, x_n^*)$$
$$= Q^* + \frac{a_r \Delta}{\sum\limits_{i=1}^{n} a_i x_i^* + a_r \Delta} \left\{ \frac{k_r + \dfrac{G\left(\sum\limits_{i=1}^{n} x_i^* + \Delta\right) - G\left(\sum\limits_{i=1}^{n} x_i^*\right)}{\Delta}}{a_r} - Q^* \right\} . \tag{58}$$

The limiting case of (57) thus leads to (55). The inequality (56) may be proved in a similar way.

Thus $(\overset{*}{x}_1, \overset{*}{x}_2, \ldots, \overset{*}{x}_n)$ is the program filled up with x^* according to the permutation $(j_1, j_2, \ldots, j_n)$ defined by

$$d_{j_1}(x^*) \leqslant d_{j_2}(x^*) \leqslant \ldots \leqslant d_{j_n}(x^*) \tag{59}$$

where the differential cost ratio functions

$$d_i(x) = \frac{k_i + G'(x)}{a_i} \qquad (i = 1, 2, \ldots, n) \tag{60}$$

take the place of the constant differential cost ratios occurring in (44) for the case of a linear $G(x)$ (constant $G'(x)$). A comparison with the case of linear $G(x)$ will immediately show that this program is also optimal for the linear problem, where the function

$$\tilde{G}(x) = G(x^*) + G'(x^*)(x - x^*) \tag{61}$$

has been substituted for $G(x)$, i.e., the linear function corresponding to the tangent of the curve $G(x)$ at the point x^* has been substituted. The problem is then the following: Let a value be found, for which the optimal programs of the linear problem according to (61) contain a program with a total output x^*.

It is not difficult to construct a recursive procedure to determine a value of x^* that will produce the result in a few steps. In choosing the initial value, it is useful to bear in mind that the firm's optimum total output x^* is always positive and may not exceed the normal capacity $\varkappa$. We also know that $x^* > 0$, because for $x_1 \to 0, x_2 \to 0, \ldots, x_n \to 0$ we obtain $Q(x_1, x_2, \ldots, x_n) \to \infty$ and thus $(0, 0, \ldots, 0)$ cannot be an optimal program. On the other hand, the linear problem of (51) obviously makes sense only if $\tilde{G}(x) \geqslant 0$ so that $\tilde{G}(0) = G(x^*) - x^*G'(x^*) \geqslant 0$, i.e.,

$$g(x^*) = \frac{G(x^*)}{x^*} \geqslant G'(x^*) \,. \tag{62}$$

This, however, is only valid in the interval $(0, \varkappa)$ (see equation (7)).

4. SOME PROBLEMS OF PRICE REGULATION

In describing the model, it was postulated in Assumption 8 that prices are given. This is in fact the case in Hungarian industry from the point of view of the firm. The situation is, however, quite different from the point of view of the price authority which fixes the prices. Precisely because the firm is interested in profit, its decisions are obviously affected by prices.[8] The price authority must therefore, as far as possible, fix prices in such a way that the reactions of the firm should correspond to the purposes of general economic policies.

The authors' investigation, after clarifying the behaviour corresponding to the firm's interests with the various types of incentive systems, also used the mathematical apparatus, so far developed, to deal with problems of price regulation. Only three of these problems will now be singled out for discussion.

(i) In the discussions in Hungary it has been proposed to introduce so-called *indifferent* prices for practical use. These are supposed to be prices which give the enterprise no reason to prefer producing and selling one article rather than any other. In other words, in a given system of incentives, and with a defined total output for the firm, the price system leads to indifference if the index of profitability of the firm does not change, whatever the composition of articles in the feasible program satisfying the given total output.

Let us examine how indifferent prices should be determined. Let the

[8] It is a different question to what extent the effect of prices is limited by other factors (e.g., the instructions of superior bodies). This has fallen outside the purview of the present investigation.

indifferent price of the ith article be $\bar{a}_i$. Transform the expressions (28) and (40) of the objective functions to the forms:

$$P(x_1,x_2,\ldots,x_n) = \frac{\sum_{i=1}^{n} x_i\left\{b_i \sum_{r=1}^{n} x_r - G\left(\sum_{r=1}^{n} x_r\right)\right\}}{\sum_{i=1}^{n} x_i} \tag{63}$$

and

$$Q(x_1,x_2,\ldots,x_n) = \frac{\sum_{i=1}^{n} a_i x_i \dfrac{k_i + g\left(\sum_{r=1}^{n} x_r\right)}{a_i}}{\sum_{i=1}^{n} a_i x_i} . \tag{64}$$

In both cases the objective function is the weighted arithmetical mean value of functions which depend on the program $(x_1, x_2,\ldots, x_n)$ only through the total output $x = x_1 + x_2 + \ldots + x_n$.

In the case of a *sum incentive* system it follows directly from (63) that, for a given total output x, the profit $P(x_1, x_2,\ldots, x_n)$ can only be independent of the composition of articles in the program if

$$b_1 x - G(x) = b_2 x - G(x) = \ldots = b_n x - G(x) . \tag{65}$$

Hence the formula for the indifferent price is

$$\bar{a}_i = k_i + \beta \qquad (i = 1,2,\ldots, n) , \tag{66}$$

where β is an arbitrary (possibly negative) number, for which all a_i's are still positive.

It may be seen from formula (66) that if the price system is indifferent for a particular value x of total output, then it will also be indifferent for any value of x, for x does not appear in (66). The situation is different in the case of a *ratio incentive* system. An argument similar to the preceding one will show that a price system is indifferent for a particular value x of total output if

$$\frac{k_1 + g(x)}{a_1} = \frac{k_2 + g(x)}{a_2} = \ldots = \frac{k_n + g(x)}{a_n} , \tag{67}$$

i.e., if

$$a_i = \gamma(k_i + g(x)) \qquad (i = 1,2,\ldots, n) . \tag{68}$$

Here, therefore, the price formula depends also on the total output x. It can be noticed, however, that if $x \neq \varkappa$ (the normal capacity), then it follows from the inequality

$$g(x) > g(\varkappa) = g_{\min} \tag{69}$$

(a consequence of (6)) that a feasible program with total output $\varkappa$ will be determined instead of the original feasible program with total output x.

The new program with total output $\varkappa$ will have the same ratios of individual outputs as the program with total output x, but the cost ratio will certainly be smaller. Indifference is thus also logically equivalent to fixing the total output at the normal capacity level. Thus the formula for an indifferent price is

$$\bar{a}_i = \gamma(k_i + g_{\min}) \qquad (i = 1, 2, \ldots, n)\,, \tag{70}$$

where γ is an arbitrary constant.

The price formulae that would render the price system truly indifferent if they were strictly complied with may thus be *theoretically* determined. In *practice*, however, prices will inevitably fail to satisfy the required conditions even if originally determined according to these formulae. Prices are fixed in Hungary for several years, and it is probable that the individual costs of various articles will not change in equal measure after approval of the price; occasional changes in wages will not affect the costs of different articles in equal measure; etc. The indifferent price cannot thus become a part of actual practice. The introduction of profit-sharing incentives inevitably involves a selection between articles by the firm according to considerations of profitability.

(ii) In some cases the economic administration wishes to use the tools of price policy to induce the firm to manufacture certain articles rather than others. There have been instances of this in past practice. Thus, for example, in order to encourage the production of cotton goods that are particularly suitable for export, the foreign trading firms pay the manufacturing firm a surcharge.

In the case of sum incentives, a system of prices can be developed according to which the preference scale of the *firm* regarding the composition of output coincides completely with the *social* preference scale considered desirable. The task is the following:

The price authority determines the individual unit costs k_i of the articles (on the basis of the calculations of the firm) and in fixing the price, permits the addition of a price margin b_i. This must cover the overall costs as well as the firm's profits. The values of the b_i's must be chosen so that the socially preferred articles always carry larger price-margins. In this case the firm will follow this order of articles in making up its program.

In the case of ratio incentives, the problem is more complicated. The following formulation of the task must suffice:

What formulae should determine prices in order to ensure that the firm always gives preference to one group of articles, the *preferred articles* as against the remaining, *nonpreferred articles*? Let the maximal value of the function $G'(x)$ within the interval $0 \leqslant x \leqslant \varkappa$ here be called $G'_{\max}$. This is the greater of the values $G'(0)$ and $G'(\varkappa)$. Let $d'_{\min}$ be the minimal value of the differential cost ratios of the nonpreferred articles at $x'_{\min}$. Let the individual unit cost of the preferred ith article be k_i^*. It may then be proved that if the unit price a_i^* of the preferred article be determined by the price authority according to the formula

$$a_i^* > \frac{k_i^* + G'_{\max}}{d'_{\min}} \qquad (i = 1, 2, \ldots, n)\,, \tag{71}$$

then it is always worthwhile for the firm to give preference to the preferred article.

(iii) One of the problems of price regulation by an authority is the size of the profit, for the authority generally endeavours to regulate the difference between the unit price and the whole unit cost more or less uniformly.

In the case of sum incentives, the price authority can influence the firm's total output by determining the magnitude of the profit, for it is easily seen that if the firm is producing according to an optimum program, the following rules will be followed:

(1) If the unit prices are greater than the minimal unit cost (if the articles are profitable), then the output is greater than the normal capacity. The more the unit prices surpass the minimum unit costs, the more will the output exceed the normal capacity. To show this, let us write the profitability conditions

$$a_i \geqslant k_i + g_{\min} \qquad (i = 1, 2, \ldots, n) \tag{72}$$

in the form

$$b_i \geqslant g_{\min} \qquad (i = 1, 2, \ldots, n) \tag{73}$$

using the relation (14). Since the graph of the function $B^{*\prime}(x)$ can then only intersect the graph of $G'(x)$ beyond $\varkappa$, two conclusions may hence be drawn:

First, because $\varkappa > \bar{x}_{\min}$, the output corresponding to the abscissa of the point of intersection is the optimal one.

Secondly, the optimal output is greater or, at the least, as great as the normal capacity (Figure 8).

One of the main purposes of the general price adjustments of January, 1959 was everywhere to eliminate prices which were lower than the actual average unit cost, while at the same time ensuring that profits should not be too great. In the case of a sum incentive system, these principles also involve decisions about outputs. Their effect is to ensure that the outputs should somewhat—though not by a great deal—exceed the normal capacity.

(2) If unit price is less than the minimum unit cost (if an article involves a loss), but its price margin is still positive, then output is less than normal

capacity and may even be zero. Reformulating the relevant conditions, it may only be stated that

$$0 < b_i < g_{\min} \qquad (i = 1, 2, \ldots, n)\,, \tag{74}$$

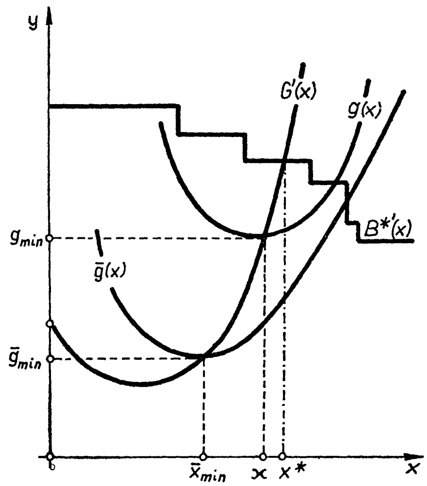

FIGURE 8

i.e., that the abscissa of the point of intersection of the two graphs—if indeed they intersect at all—is less than the normal capacity. It may happen that the abscissa is less even than $\bar{x}_{\min}$ so that the optimal output is even smaller, and may possibly be 0 (see Figures 9 and 10).

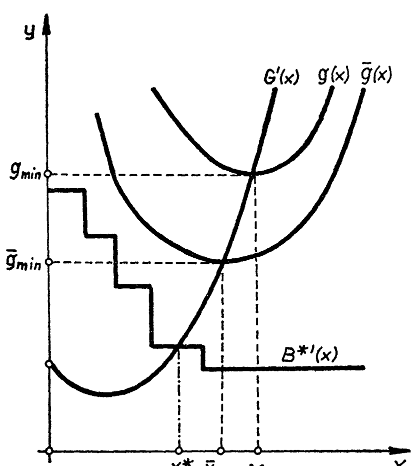

FIGURE 9

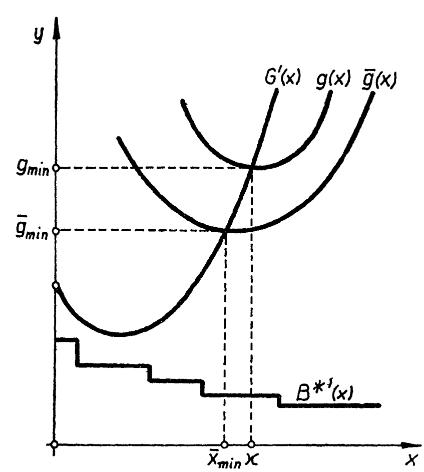

FIGURE 10

(3) If unit prices are not greater than the corresponding individual costs, then it is *certainly* not worthwhile for the firm to produce. For then

$$b_i \leqslant 0 \qquad (i = 1,2,\ldots, n)\,, \tag{75}$$

i.e., the two graphs are certain not to intersect.

In the case of ratio incentives, the relations are of an entirely different nature. Limitations of space do not permit them to be considered here.

5. THE CHOICE OF ALTERNATIVE INCENTIVE SYSTEMS

The authors were unable to recommend a solution relying uniquely on either of the two systems of incentives. Instead, conclusions were deduced showing the economic policies for whose promotion sum incentives are appropriate and those for which ratio incentives are more suitable. The choice must always be made in accordance with the economic policies held valid for a particular sector or branch of industry.

When deciding on which type of incentive system to apply, the economic administration may consider several criteria:

First criterion: What total output do the two types of incentives encourage in the case of short-run decisions?

In the case of sum incentives, the firm's total output is not stabilized so much as with ratio incentives. The firm's total output may, as has been seen in the section on price regulation, be either more or less than the normal capacity, depending on the prices. What is more, in the case of certain prices, as has been shown, the least unfavorable course for the firm, from the point of view of profitability, is not to produce at all. The use of sum incentives is desirable if economic policy requires the firm's total output to be raised above the normal capacity, or if the economic administration wishes to make use of the price system to influence the firm's total output.

Sum incentives are expressly harmful if economic policy is aimed at preventing progressive costs from arising, or if the articles are, in the given situation, unavoidably produced at a loss, while there is no way of changing the price and their production is nevertheless necessary. For, in this case, a situation may arise where it is in the interest of the firm considerably to decrease and perhaps even to stop production.

Ratio incentives set a limit to output. The firm never raises total output above normal capacity and often stays under it. On the other hand, it is worthwhile for the firm to produce, whatever prices are; the cessation of production can never be the optimum decision for the firm (cf. the last paragraph of Section 3).

Ratio incentives are desirable if economic policies require that output be restrained in a particular sector. If this is the case, then the economic administration may use this method also to prevent the firm from raising its output above the normal capacity.

Second criterion: Which type of incentive system makes programming easier? This criterion must be considered when the economic administration expressly requires that firms draw up programs according to profitability considerations.

Programming is much easier in the case of sum incentives than with ratio incentives. In the first case, the problem may be solved by a relatively simple ordering, while in the second, the solution may only be approximated by the iterative application of similar methods.

Third criterion: If the economic administration wishes to make use of instruments of price policy to influence the firm's programming (e.g., if it wishes to encourage the firm to give preference to socially more important articles), then which system of incentives will enable it to do this better?

Both systems provide an opportunity to do this; but, as has been seen in Section 4, subsection (ii), it is easier and simpler to implement this objective in the case of sum incentives.

There are also other criteria which the economic administration must consider when deciding which form of incentive system to apply. These have been analysed, but they will not be discussed in this brief paper.

The authors do not believe that the concrete programming procedures that have been indicated in this paper will be widely applicable in any direct way to other tasks. The model on which these investigations were carried out bears the marks of the peculiar economic features of Hungarian Socialist industry. This paper has rather been intended to give an idea of the ways in which the problems of the profits and costs of firms are presented to the economists of a Socialist country and how mathematical methods are used in Hungary to elucidate some of the characteristic problems of a planned economy.

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