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THE NORMAL STATE OF THE MARKET IN A SHORTAGE ECONOMY: A QUEUE MODEL

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Abstract

The subject of this study is an economy characterized by chronic shortage and queuing. The paper elaborates a very simple model for a single good, primarily intended as an illustration of an analytical framework for studies of shortage phenomena. Our main concern is to describe a market which is away from Walrasian equilibrium, and nevertheless in a stationary state, permanently restoring its basic properties. Central concepts in our analysis are such non-financial costs of shortage as queuing time, postponement of purchase and forced substitution.

I. Introduction

The subject of our study is an economy characterized by chronic shortage and queuing (seller's market). This is the case in many sectors of the Eastern European socialist economies. The phenomena of shortage may appear in other systems as well: in housing markets under rent controls, in the health services of some developed capitalist countries, or in the consumer-goods markets of the developing countries.

There is a growing interest in the theory of an economy out of, or even far away from, Walrasian equilibrium; see for example the works by Clower (1965), Barro & Grossman (1971, 1974), Benassy (1975), Malinvaud (1976) and McCafferty (1977). We would like to contribute to this course of research. The subject is very extensive and has many facets.[1] This article touches only on a

[1] One of the authors, J. Kornai, has been engaged in the study of shortages for a long time. The antecedents of the present research are his works (1971), (1974) and (1977). In 1977 he gave a lecture course at Stockholm University under the title "Economics of Shortage". Work is in progress to develop the lecture notes into a book, which will discuss many different aspects of the theory of shortage. The research carried out, jointly with J. W. Weibull, and reported in this and a forthcoming second paper, branched out from this broader study of the economics of shortage.

J. Kornai takes the opportunity to express his gratitude to the Institute for International

few points, focusing mainly on some microeconomic aspects. Our main concern is to describe a market which is away from Walrasian equilibrium—and nevertheless in a stationary state, permanently restoring its basic properties. Appraising our own results: although we arrive at a few theorems, we would like to draw attention more to the *analytical framework*, i.e. to the specific points of view from which we consider the functioning of a market in chronic shortage.

This paper elaborates on a very simple model, to introduce the reader into our framework. Later on, we will publish papers where we relax some of the most restrictive assumptions and reflect more on the complexities of the problem.

One comment should be made here concerning deterministic versus stochastic modelling of queue systems. In contrast to other queue models, which are usually stochastic, the present model is deterministic. This choice of approach reflects our belief that in situations characterized by chronic shortage, the stochastic element is secondary in comparison to the interdependencies and feedback mechanisms regulating the system. Although a general model should include stochasticity as well, some of the fundamental relationships may be explained within a deterministic framework.[1]

One more preliminary comment. What we present here is a *descriptive* theory. We do not raise normative questions. Shortages and queues are facts of life. We neither praise nor condemn—we try to understand them.

II. The Model: General Description

The model is a deterministic stock-flow model, formalized in terms of a system of ordinary differential equations. We describe it in two stages. In Section II, we discuss the model in a rather general way, more in qualitative and micro-oriented terms, for expository purposes and for interpretation. In Section III, we go into the technical details and present the full formal description.

II.1. *The Structure of the Market*

We study a market trading a single *good G*. This can be a specific commodity, or an aggregate composed of different commodities. The good is traded as indivisible items; each buyer will acquire only one item on the purchase occasion. (For example he buys one car or one refrigerator.)

Economic Studies at the Stockholm University for support of his research and to his Swedish colleagues for many inspirations. J. W. Weibull gratefully acknowledges the support of the Swedish Council for Building Research, as well as the constructive criticism provided by his colleagues at the Department of Mathematics at the Royal Institute of Technology, Stockholm.

Both authors are indebted to Lars-Göran Mattsson, Ingemar Nåsell and Johan Philip for valuable suggestions and comments.

[1] We are grateful to Lars-Göran Mattsson, who originally advocated this viewpoint.

There is only one *seller* (a monopolist or the aggregate of several individual sellers).

There are *n buyers*. The whole population of buyers is divided into subpopulations called *groups* of buyers. Each group has its own characteristic reactions to the market. We call the representative member of group i *a buyer of type i*. There are k groups, and n_i members in group i; $\sum_{i=1}^{k} n_i = n$. The number of participants (one seller, $n_1, n_2, \ldots, n_k$ buyers) does not change over time.

Although the model, as it is formally defined in Section III, is deterministic, it may be viewed as a "hybrid" model, consisting of a collection of deterministic relationships between mean values for stochastic components. In particular, we focus on a sequence of decisions made by the buyers when shopping. At each such decision stage, we model the aggregate behavior of the buyer groups in terms of *flows*: the inflow of buyers into the decision stage and the flow shares out of the decision stage, corresponding to each decision alternative (there are only two options at each decision stage). However, these deterministic flow shares may be regarded as mean values for stochastic individual choice behavior, the shares being identified as choice probabilities. At other places in the model, we talk about deterministic *rates*. As in the case of the flow shares, these rates may be interpreted as mean values for stochastic individual behavior. In order to illustrate the deterministic model assumptions, we frequently make such micro-oriented, stochastic interpretations. In the literature on stochastic queue models, our approach is sometimes referred to as "the fluid approximation"; see e.g. Kleinrock (1976).

II.2. *The Shopping Algorithm*

Shopping is a dynamic process, a series of decisions. Since shopping has some behavioral regularities, the process can be described as an *algorithm*. The structure of such an algorithm may of course be different for different shopping situations. In the following analysis, we focus on one particular algorithm that we feel has some of the ingredients of real-life situations and yet is analytically tractable. We illustrate the shopping process in Fig. 1 in the form of a block diagram.

We accompany an individual buyer of type i on his shopping route. He departs from the field called START.

His first decision problem is the following. Should he try to buy good G, traded on the market of our model? Or rather, should he buy the *substitute good H*, traded on another market (outside the scope of our model)? Good H can be a specific close substitute or a composite good, representing the aggregate of all close and distant substitutes for G. We assume that income and all other factors influencing the buyer's decision are given and that they are invariant over time. The only signals to be considered at this decision stage are the prices, i.e. the *price ratio* $\pi = p_G/p_H$.

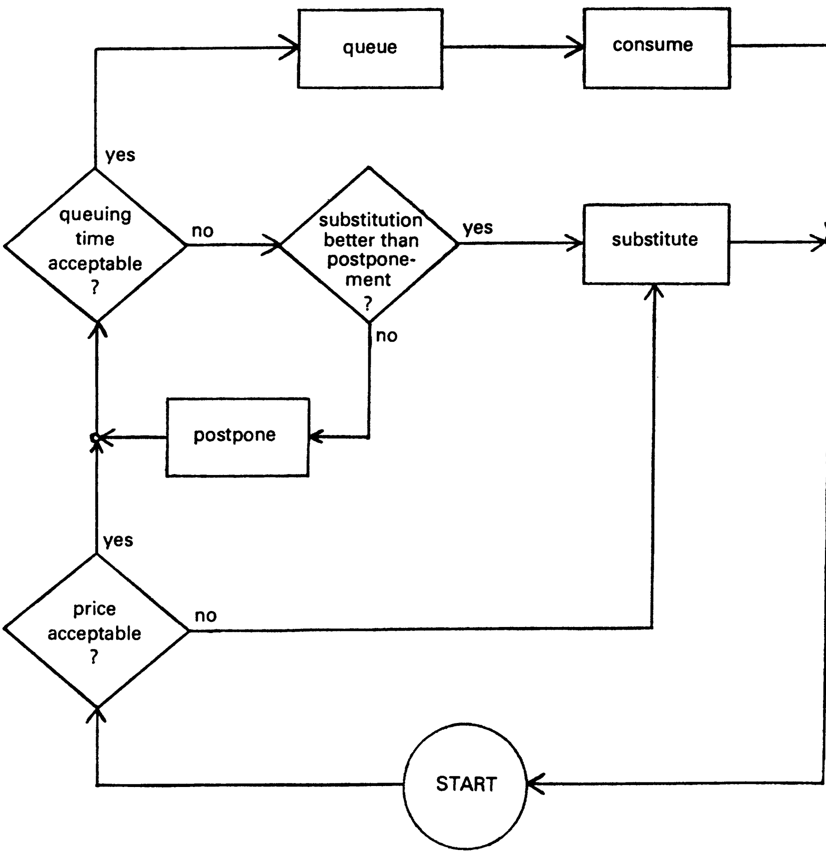

Fig. 1. The shopping algorithm.

We denote the buyer's *initial buying propensity* by $a_i(\pi)$. This is a non-increasing (usually decreasing) function of π. Given the price ratio π, $a_i(\pi)$ is the fraction of buyers of type i that will decide to look for good G rather than good H. In micro-oriented terms $a_i(\pi)$ may be interpreted as the probability that the buyer will initially prefer good G to good H.

So far, we have an orthodox point of departure. The function $a_i(\pi)$ is an ordinary demand function, depending on relative prices; only the technical form is different from the usual, since we need this particular form for our further analysis.

Keep in mind the adjective "initial". It refers to the fact that $a_i(\pi)$ represents an original buying intention at the beginning of the shopping route —subject to later revision, after encountering the shortage phenomena. It expresses a *hypothetical* demand, assuming the absence of shortage.[1] The

[1] Our concept of "hypothetical demand" is closely related to—although not identical with—Clower's concept of "notional demand". The Clowerian concept is used in a comparative static framework; we describe demand formation as a dynamic process. In Clower's model actual transaction cannot exceed notional demand, whereas we assume

fraction $a_i(\pi)$ of the shopping members of group i are willing to spend their money on G, *if* the good G were available on the supply side without delay.

Our buyer proceeds to the seller's place. There is a queue. He is hesitant: should he join the queue? We assume that there is only one factor influencing his decision, and that is the expected *queuing time* w. The larger the w, the more reluctant the buyer is to join the queue. The quantity $f_i(w)$ expresses the *queuing propensity*. This means that out of the total number of type i buyers, interested in purchasing good G, a fraction $f_i(w)$ will join the queue, and the rest, i.e. the fraction $(1-f_i(w))$, are not willing to join the queue right now.

Let us assume for a moment that our buyer belongs to the first subgroup, and he enters the queue. He waits, patiently or impatiently, until served, and then he goes home with the newly-acquired good. We suppose that after some time has passed, his desire to acquire another item of good G or H arises—and the whole process begins again. The reasons for renewal of the need will not be discussed. (For example, the buyer uses up the good or it becomes obsolete or out of fashion after a while.) Anyway, out of the total number of G-satisfied buyers at time t, we assume that the fraction $\gamma_i \cdot dt$ will exhibit a need to acquire another item (of either G or H) in the infinitesimal time interval $(t, t+dt)$. The assumption that this fraction is independent of the time t is made here for technical convenience. In micro-oriented terms this assumption implies that the satisfaction time, i.e. the time interval from the date when a buyer acquires an item of good G until the date when the renewed need arises, considered as a random variable, has an exponential probability distribution with mean value $1/\gamma_i$. The quantity γ_i will be called the *(post G) need renewal rate* and $1/\gamma_i$ will be referred to as the *average G-satisfaction time.*

We now turn to the other branch of the algorithm, to the buyer who was scared away by the long queuing time. He has different options. He may insist on buying good G, but he postpones the decision of whether or not to join the queue to a later time. This can be a reasonable action in the case of a real "physical" queue: people standing in front of the butcher's shop in the morning, or sitting in the doctor's waiting room.[1] The buyer's behavior is described by two attributes. First, there is a *postponement propensity* denoted by b_i. And second, there is a postponement time. After that time has elapsed,

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that the buyer may apply forced substitution. He may buy *more* of good H than his hypothetical demand, if waiting time for good G is unacceptably long.

Studying empirical evidence of chronic shortages, we think our assumption is less restrictive, and renders more generality to the description of buyer's behavior.

To avoid terminological confusion, we use different names for these two—partly overlapping, but partly differing—phenomena.

There are some more related, but not strictly identical concepts in our terminology and in the language of the "disequilibrium school". Lack of space does not allow detailed comparison of the conceptual frameworks.

[1] It is not a reasonable option in the case of a "notional" queue, e.g. when serial numbers are distributed among the members of the queue, after which everybody may go home and return when called.

the buyer returns and considers again: should he join the queue? Here we make a similar assumption as for the G-satisfied buyers, i.e. in the infinitesimal time interval $(t, t+dt)$, the fraction $\varrho_i \cdot dt$ of the total number of postponers in group i will return to reconsider their joining the queue. In micro-oriented terms: the postponement time is an exponentially distributed random variable with mean value $1/\varrho_i$. We call ϱ_i the *reconsideration rate* and $1/\varrho_i$ the *average postponement time*.

Another option for those who did not join the queue, but do not postpone the same decision, is to substitute good H for good G. We call this *forced substitution*, forced by shortage and revealed in the unacceptably long queuing time. There were some people, the fraction $(1-a_i(\pi))$ of customers of type i, who made a *voluntary substitution*, considering exclusively the relative prices of G and H. But now some more substitutioners follow them, on an involuntary basis. Considering relative prices, they would prefer G to H—but threatened by the long queuing time they would rather revise their initial demand and are willing to accept good H instead of good G. Forced substitution is the key phenomenon in understanding what is happening under chronic shortage. We denote the *forced substitution propensity* by $c_i(\pi)$. (As in the case of the initial buying propensity, we assume that the forced substitution propensity depends only on the relative price π.)

The third option is to give up the purchase of both G and H, and simply keep the money unspent. This can be called *shortage forced saving*.[1]

Aware of all these options, we introduce some drastic simplifications in the present expository model. We exclude the possibility of shortage forced saving and assume the following. If the buyer is neither willing to join the queue for G immediately, nor willing to postpone the same decision, then he must be willing to accept forced substitution and to buy good H. This good is always available immediately. One possible interpretation of our assumption is that good H represents the composite commodity "goods other than G". Even in the worst degree of shortage there is always *something* in the stores. Most of the buyers are willing to spend their money on something, somehow. This is a rather realistic assumption for a very large part of buyers' decisions in a shortage economy.[2] Our assumption is represented by the following relationship: $b_i + c_i = 1$. For the sake of simpler notation, we use only the term c_i and the postponement propensity will be denoted by $(1-c_i)$.

In the case of acquiring good H (due to voluntary or forced choice) the buyer goes through a similar satisfaction time as in the case of good G. In particular, we assume that in the infinitesimal time interval $(t, t+dt)$, the fraction $\varkappa_i \cdot dt$ of all H-satisfied buyers of type i will exhibit a need to acquire

[1] The first option, postponement of the decision to join the queue for G, may imply temporary forced saving as well.

[2] A detailed discussion of forced saving will follow in other publications resulting from our research.

another item (of either good G or H). We call $\varkappa_i$ the (*post H*) *need renewal rate* and $1/\varkappa_i$ the *average H-satisfaction time.*

This brings us to the end of the whole cycle.

II.3. *The Buyers' Attitude*

To sum up, the *buyers' attitude* is characterized by the following collection of functions and parameters:

$a_i(\pi)$	= initial buying propensity at relative price π
$f_i(w)$	= queuing propensity at queuing time w
$c_i(\pi)$	= forced substitution propensity at relative price π
$(1-c_i(\pi))$	= postponement propensity at relative price π
$\gamma_i, \varkappa_i$	= (post G) and (post H) need renewal rates, respectively
ϱ_i	= reconsideration rate.

The functions and parameters listed above describe the attitude of group i. It may be noted that the attitude (considered as a vector) is specified in terms of only two "signals": the relative price π and the queuing time w. Moreover, we have separated the buyers' considerations of these two signals into successive, distinct decision points in the shopping algorithm. Thus, having once accepted the price, the buyers consider the queuing time without regard to the price. (Conjoint consideration of price and queuing time may be analyzed without technical difficulties.)

A brief comparison with the usual market models is now in order. As already mentioned, we go along with the traditional description at the first step of the algorithm—the demand function depends on relative prices. The usual model terminates here, with the tacit assumption that it is enough to know what the buyer's intention is. If he wants to buy a given quantity at the price asked for by the seller—he will surely get it. We admit that this tacit assumption is more or less legitimate when excess demand is only exceptional and temporary. It can be applied to the description of a market where automatic forces rapidly eliminate excess demand. But under the circumstances of *chronic* shortage the same tacit assumption becomes unjustified; the description of the buyer's behavior must not stop here. What will happen after the first step, i.e. after determining the initial demand? In an economy where excess demand is exceptional, buying can be treated as a one-stroke action: a decision regarding purchase intention and actual purchase can be condensed into one moment of time. On the other hand, in a shortage economy, buying may be described only as a process over time, looking at the original decision and later on at the revisions, choice between further options, etc. Therefore, the following dilemmas were introduced into the model: joining the queue, postponing, and accepting forced substitution. (In a forthcoming paper, we will discuss the case of searching buyers.)

II.4. *The State Variables of the Buyers*

At any fixed time t, each buyer plays exactly one of four different roles. The number of buyers in each role will be represented in the model by the following four *state variables*:

$x_{1i}(t)$ = the number of buyers of type i who are queuing at time t; briefly: the *queuing buyers;*

$x_{2i}(t)$ = the number of buyers of type i who have earlier acquired an item of good G and, at time t, are not yet ready to start the shopping process again; briefly: the *G-satisfied buyers*;

$x_{3i}(t)$ = the number of buyers of type i who have earlier acquired an item of good H and, at time t, are not yet ready to start the shopping process again; briefly: the *H-satisfied buyers;*

$x_{4i}(t)$ = the number of buyers of type i, who have earlier postponed the decision to join the queue, and at time t do not yet reconsider the same decision; briefly: the *postponing buyers.*

$$x_{1i}(t)+x_{2i}(t)+x_{3i}(t)+x_{4i}(t)=n_i \quad i=1, \ldots, k, \quad t\geqslant 0.$$

$$x_j(t)=\sum_{i=1}^{k} x_{ji}(t) \qquad j=1, 2, 3, 4, \quad t\geqslant 0.$$

In the following analysis, all the variables above are treated not as integers but as real numbers. At any time $t\geqslant 0$, the vector $(x_{11}(t), \ldots, x_{1k}(t), x_{21}(t), \ldots, x_{2k}(t), x_{31}(t), \ldots, x_{3k}(t), x_{41}(t), \ldots, x_{4k}(t))$ will be called the *state* of the buyer population at time t. Conversely, any non-negative real vector $(x_{11}, x_{12}, \ldots, x_{4k})$ satisfying $x_{1i}+x_{2i}+x_{3i}+x_{4i}=n_i$ for all i, will be called a *feasible state* for the buyer population.

II.5. *Service Capacity and Effective Service Flow*

In subsections II.2–II.4 we discussed the buyers. We now turn to the description of the seller. In the context of the present model, supply will be represented by service capacity and trade by effective service flow.

The *service capacity* of the seller is denoted by λ. This is the maximal number of buyers who can be served per unit of time. In the case of a store, λ depends on the initial inventories and on the deliveries of supplies to the store. In the case of a productive firm, λ depends on the initial inventories and on the production capacity. We disregard inventories and assume that λ is time-invariant and fixed exogenously.

As the queue length x_1 is treated here as a continuous variable, it would be natural to let the *effective service flow*, i.e. the actual number of buyers served per time unit, equal λ for $x_1>0$ and zero for $x_1=0$. In other words, full service as long as there is a queue, and no service if there is no queue (the person being served is included in the queue). However, this "switching rule" type of dependency of the effective service flow (s) on the queue length (x_1) is discontinuous at $x_1 = 0$, and such a discontinuity would be technically disturbing in the analysis of the dynamics of the system of buyers. Therefore, we replace this discontinuous relationship by a continuous relationship along with a limit argument. More precisely, first we let the effective service flow (s) depend on the queue length (x_1) according to the following equation:

$$s(x_1(t)) = \lambda \cdot h_\sigma(x_1(t)), \tag{2.1}$$

where h_σ is a continuous function that increases from zero to unity on the interval $[0, \sigma]$ and equals unity on the interval $[\sigma, +\infty)$. The parameter σ, called here the "smoothing coefficient", is assumed to be a small, positive constant. Later on in the analysis, we let σ approach zero, thereby letting the continuous relationship (2.1) approach the original, discontinuous "switching rule".

II.6. *The queue*

All actions of the seller and of the buyers are mutually independent—with one exception. The only place of interaction is the queue. This is where they meet; the queue is the linkage which makes the participants of the system mutually interdependent. The queue might be either "physical", i.e. consisting of persons waiting in a waiting-room or a shop, or "notional", i.e. consisting of a pile of requests or orders at the seller's office.

The queuing time w appears as the argument in the queuing propensities $f_i(w)$, $(i = 1, 2, ..., k)$, which does not mean that presumptive customers necessarily perceive w correctly—only that their aggregate behavior is a function of w. We consider the case of queues without priorities, i.e. a newcomer joining the queue has to wait the time it takes to serve all customers standing ahead of him in the queue. This motivates the following relationship[1]:

$$w(t) = x_1(t)/\lambda \tag{2.2}$$

It should be noted that this equation may also be used as an approximation in some cases with more than one queue for good G. If there are many queues and the arriving customers always choose the queue with the shortest queuing time, then the queuing times in the different queues will tend to become equal, and equation (2.2) applies to the aggregate of queues.

The queue is comprised of members from the different buyer groups. In general, these groups may be more or less well-mixed in the queue. However, for analytical tractability we assume that they are homogenously mixed. Let $s_i(t)$ denote the outflow of served buyers of type i at time t:

$$s_i(t) = \begin{cases} \dfrac{x_{1i}(t)}{x_1(t)} \cdot s(x_1(t)) & \text{if } x_1(t) > 0 \\ 0 & \text{if } x_1(t) = 0 \end{cases} \tag{2.3}$$

[1] For $\sigma < 1$, the quantity $x_1(t)/\lambda$ equals the time a newcomer at time t has to wait for his service to start.

In other words we assume that the outflow of served buyers of type i from the queue is proportional to the share of such buyers among all buyers in the queue. For an initial or transient state of the buyer population, this may indeed be a crude approximation (the queuing members from one buyer group may e.g. stand ahead of all other queuing buyers). In a stationary state, however, the homogeneity assumption is appropriate, granted independent individual behavior.[1] The quantity s_i will be referred to as the *effective service flow of buyers of type i* ($i=1, 2, ..., k$), $s=s_1+s_2+...+s_k$.

III. The Model: Formal Summary

After explaining the institutional and microeconomic implications of the model, we are ready to summarize the formal description with some repetition of Section II.

III.1. *Exogenous Data*

The following *parameters* are assumed to be exogenously given, fixed real numbers: λ, π, γ_i, $\varkappa_i$, ϱ_i ($i=1, 2, ..., k$). Let R_+ denote the set of non-negative real numbers and [0, 1] the closed unit interval. The following *functions* are assumed to be exogenously given, fixed and defined on R_+ and take values in [0, 1]: f_i, a_i, c_i, h_σ ($i=1, 2, ..., k$).

Our main assumptions are implied in the qualitative properties of the model, discussed in Section II. We now give a partial summary of the assumptions, listing only those which are needed for the mathematical specification of the exogenous parameters and functions. Some of these assumptions, listed below, are mere repetitions of earlier verbal formulations. Some others will be introduced here. (Observe that the assumed properties of the functions are taken to hold in the whole domain R_+ of the functions.)

A1: The parameters λ, γ_i, $\varkappa_i$ and ϱ_i ($i=1, 2, ..., k$) are all positive. $\varrho_i > \varkappa_i$ for all i. The parameter π is non-negative.

A2: The functions f_i ($i=1, 2, ..., k$) are all non-increasing with $f_i(0)=1$ and have continuous first derivatives f_i'.

A3: The functions a_i ($i=1, 2, ..., k$) are all non-increasing and continuous with $\lim_{\pi\to\infty} a_i(\pi)=0$.

A4: The functions c_i ($i=1, 2, ..., k$) are all non-decreasing and continuous. If $a_i(\pi)=0$ for some i and π, then $c_i(\pi)>0$.

A5: The function h_σ ($\sigma>0$ being fixed) is increasing on the interval $[0, \sigma]$. Moreover, h_σ is twice differentiable and satisfies $h_\sigma(0)=0$ and $h_\sigma(x)=1$ for every $x \geqslant \sigma$.

[1] Assumption (2.3) is disturbing from a logical point of view. Namely, in the case of more than one buyer group it may conflict with the interpretation of (2.2) in terms of a strictly ordered queue. An alternative interpretation of (2.2), which is consistent with (2.3), is that the members of the queue are drawn at random for service. Assuming an equal chance of being drawn and average service time per buyer $1/\lambda$, equation (2.2) gives the expected queuing time and (2.3) the average service flows for the different groups.

These assumptions require a few comments.[1] First, In A1 we assume that the average H-satisfaction time $(1/\varkappa_i)$ exceeds the average postponement time $(1/\varrho_i)$. In other words we think of situations with "short-term" postponement, as compared to consumption time.

Second, in A4 we assume that if the relative price is so high that type i buyers have zero initial buying propensity, then they have positive propensity for forced substitution.

Third, the smoothing function h_σ requires a comment. In the subsequent analysis we first derive results for an arbitrary smoothing function h_σ with $\sigma > 0$ fixed. We then let σ decrease to zero and establish results for this limiting case (to be distinguished from the case $\sigma = 0$).

III.2. *Dynamic Relationships*

As was indicated above, we will describe the dynamic evolution over time of the population state variables $x_{1i}(t)$, $x_{2i}(t)$, $x_{3i}(t)$ and $x_{4i}(t)$, $i = 1, 2, ..., k$, in terms of a system of (non-linear) ordinary differential equations. The system is given below (for $i = 1, 2, ..., k$):

$$\dot{x}_{1i} = a_i \cdot f_i(w) \cdot (\gamma_i \cdot x_{2i} + \varkappa_i \cdot x_{3i}) + f_i(w) \cdot \varrho_i \cdot x_{4i} - s_i \tag{3.1}$$

$$\dot{x}_{2i} = s_i - \gamma_i \cdot x_{2i} \tag{3.2}$$

$$\dot{x}_{3i} = [1 - a_i + a_i \cdot c_i \cdot (1 - f_i(w))] \cdot (\gamma_i \cdot x_{2i} + \varkappa_i \cdot x_{3i}) + c_i \cdot (1 - f_i(w)) \cdot \varrho_i \cdot x_{4i} - \varkappa_i \cdot x_{3i} \tag{3.3}$$

$$\dot{x}_{4i} = a_i \cdot (1 - c_i) \cdot (1 - f_i(w)) \cdot (\gamma_i \cdot x_{2i} + \varkappa_i \cdot x_{3i}) + (1 - c_i) \cdot (1 - f_i(w)) \cdot \varrho_i \cdot x_{4i} - \varrho_i \cdot x_{4i} \tag{3.4}$$

Here all state variables as well as the service rates and queuing time are functions of time, $x_{1i} = x_{1i}(t)$, etc. The service rate s_i is defined in equations (2.1) and (2.3), and the queuing time w is defined in equation (2.2). The terms "a_i" and "c_i" are abbreviations for "$a_i(\pi)$" and "$c_i(\pi)$", the relative price π being constant. The dot notation signifies time derivatives, $\dot{x} = \dot{x}(t) = dx(t)/dt$. Note that the sum of the time derivatives is zero, $\dot{x}_{1i} + \dot{x}_{2i} + \dot{x}_{3i} + \dot{x}_{4i} = 0$, reflecting the assumption that the number of buyers in each group is constant. Moreover, no state variable can take a negative value; for any feasible state (x_{ji}) with $x_{ji} = 0$ for some j and i, we have $\dot{x}_{ji} \geq 0$ by equations (3.1) to (3.4). Thus the solution to the system of differential equations is bounded at all times $t \geq 0$. In force of the assumed continuity of the first derivatives f_i' and h_σ', this guarantees the existence and uniqueness of the solution at all times $t \geq 0$; cf. Theorem 3.1 in Chapter I of Hale (1969).

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[1] A function f is said to be increasing (non-decreasing) if $x_1 < x_2$ implies $f(x_1) < f(x_2)$ $[f(x_1) \leq f(x_2)]$. It is said to be decreasing (non-increasing) if $x_1 < x_2$ implies $f(x_1) > f(x_2)$ $[f(x_1) \geq f(x_2)]$.

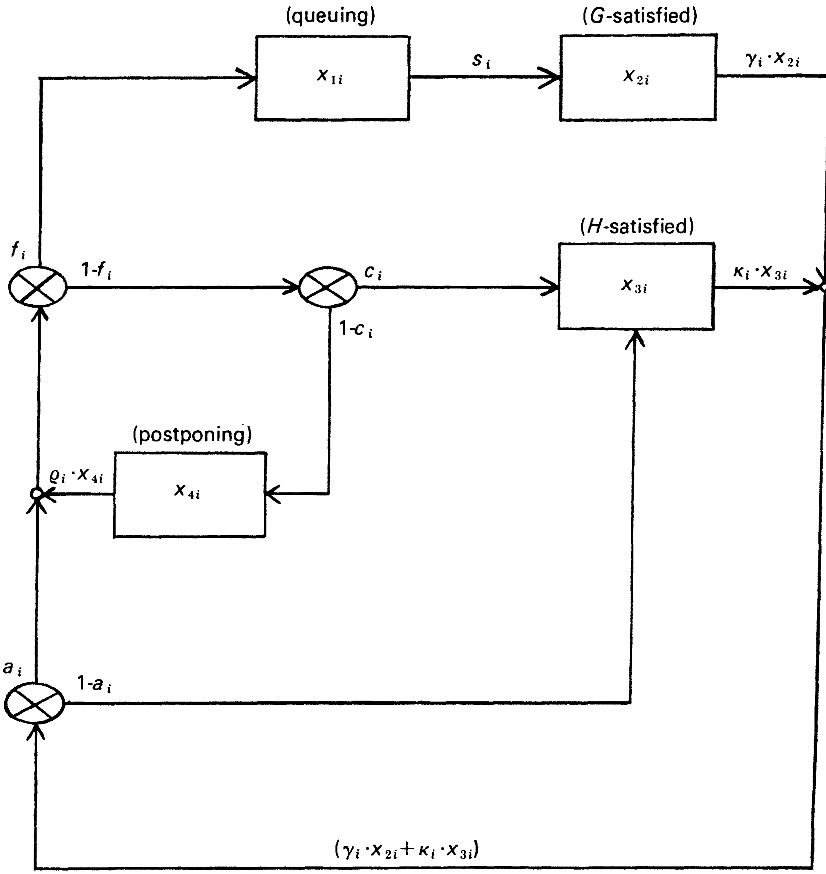

Fig. 2. Flow chart for the system of differential equations.

The system of differential equations (3.1)–(3.4) is an aggregate formulation of the description of the individual shopping behavior in Section II.2. This correspondence may be studied through a comparison of Fig. 2, which demonstrates the functioning of the system of differential equations, with Fig. 1, which demonstrates the shopping algorithm.

IV. The Normal State of the Market

The buyer population, as described by the state variables x_{1i}, x_{2i}, x_{3i}, x_{4i} $(i=1, 2, ..., k)$ is said to be in a *stationary state* if it does not change over time, i.e. if all time derivatives are zero: $\dot{x}_{1i}=\dot{x}_{2i}=\dot{x}_{3i}=\dot{x}_{4i}=0$ for all i (the system of differential equations being autonomous). In this section it is first shown that there always exists a unique stationary state. Second, for the special case of a single buyer group, it is shown that this stationary state is stable under fairly mild conditions on the queuing propensity function.

IV.1. *Existence and Uniqueness*

Proposition 1. For any collection of parameters and functions satisfying assumptions A1–A5, there exists a unique stationary state.

(Basic ideas of some of the proofs are given in the Appendix at the end of the paper.)

We now study in some detail what happens to the stationary state as we let the smoothing coefficient approach zero. The following two corollaries to the above proposition state that the queue then approaches either a positive value or zero, depending on the given collection of parameters and functions. Let

$$\phi = \sum_{i=1}^{k} \frac{\gamma_i \cdot \varkappa_i \cdot a_i(\pi) \cdot n_i}{\varkappa_i \cdot a_i(\pi) + \gamma_i \cdot (1 - a_i(\pi))} \tag{4.1}$$

and

$$A_i(w) = \frac{1}{\varkappa_i} \cdot (1 - a_i(\pi)) + \frac{1}{\gamma_i} \cdot a_i(\pi)$$
$$+ \left[\frac{c_i(\pi)}{\varkappa_i} + \frac{a_i(\pi)}{\varrho_i} \cdot (1 - c_i(\pi))\right] \cdot \left(\frac{1}{f_i(w)} - 1\right) \quad (i = 1, 2, \ldots, k) \tag{4.2}$$

$(0 < A_i(w) \leqslant +\infty)$. Furthermore, let $x^*_{1i}(\sigma)$, $x^*_{2i}(\sigma)$, $x^*_{3i}(\sigma)$ and $x^*_{4i}(\sigma)$, $i = 1, 2, \ldots, k$, denote the stationary state variable values corresponding to an arbitrary, fixed smoothing coefficient $\sigma > 0$.

Corollary 1.1. If $\lambda < \phi$, then $\lim_{\sigma \downarrow 0} x^*_1(\sigma) = x^*_1$, where $x^*_1 > 0$. Moreover, x^*_1 is the unique solution to the equation

$$\sum_{i=1}^{k} \frac{a_i(\pi) \cdot n_i}{\lambda \cdot A_i(x_1/\lambda) + a_i(\pi) \cdot x_1} = 1 \tag{4.3}$$

Let $f^*_i = f_i(x^*_1/\lambda)$. For a buyer group i with $a_i(\pi) > 0$ and $f^*_i > 0$ we have:

$$x^*_{1i} = \lim_{\sigma \downarrow 0} x^*_{1i}(\sigma) = \frac{a_i(\pi) \cdot x^*_1 \cdot n_i}{\lambda \cdot A_i(x^*_1/\lambda) + a_i(\pi) \cdot x^*_1} \tag{4.4}$$

$$x^*_{2i} = \lim_{\sigma \downarrow 0} x^*_{2i}(\sigma) = \frac{\lambda}{\gamma_i} \cdot x^*_{1i}/x^*_1 \tag{4.5}$$

$$x^*_{3i} = \lim_{\sigma \downarrow 0} x^*_{3i}(\sigma) = \left[c_i(\pi) \cdot \left(\frac{1}{f^*_i} - 1\right) + 1 - a_i(\pi)\right] \cdot \frac{\lambda}{a_i(\pi)\,\varkappa_i} \cdot x^*_{1i}/x^*_1 \tag{4.6}$$

$$x^*_{4i} = \lim_{\sigma \downarrow 0} x^*_{4i}(\sigma) = (1 - c_i(\pi)) \cdot \left(\frac{1}{f^*_i} - 1\right) \cdot \frac{\lambda}{\varrho_i} \cdot x^*_{1i}/x^*_1 \tag{4.7}$$

For a buyer group j with $a_j(\pi)=0$ and/or $f_j^*=0$ we have $x_{1j}^*=x_{2j}^*=0$, and x_{3j}^*, x_{4j}^* may be calculated directly from the stationarity conditions.

Corollary 1.2. If $\lambda \geqslant \phi$, then $\lim_{\sigma\downarrow 0} x_1^*(\sigma)=0$ and for $i=1, ..., k$ we have

$$x_{1i}^*=\lim_{\sigma\downarrow 0} x_{1i}^*(\sigma)=0 \tag{4.8}$$

$$x_{2i}^*=\lim_{\sigma\downarrow 0} x_{2i}^*(\sigma)=\frac{\varkappa_i\cdot a_i(\pi)\cdot n_i}{\varkappa_i\cdot a_i(\pi)+\gamma_i\cdot(1-a_i(\pi))} \tag{4.9}$$

$$x_{3i}^*=\lim_{\sigma\downarrow 0} x_{3i}^*(\sigma)=\frac{\gamma_i\cdot(1-a_i(\pi))\cdot n_i}{\varkappa_i\cdot a_i(\pi)+\gamma_i\cdot(1-a_i(\pi))} \tag{4.10}$$

$$x_{4i}^*=\lim_{\sigma\downarrow 0} x_{4i}^*(\sigma)=0 \tag{4.11}$$

Thus, in the limit, as we let the smoothing coefficient σ approach zero, we may distinguish between two different types of stationary states. For collections of parameters and functions satisfying the inequality $\lambda<\phi$, the corresponding stationary states approach states characterized by shortage ($x_1^*>0$), while for collections of parameters and functions satisfying the opposite inequality, $\lambda\geqslant\phi$, the corresponding stationary states approach states characterized by non-shortage. These two types of limiting states will be studied in some detail in Section V. For such a study to be meaningful, however, we have to verify the stability of the stationary state for small, positive values of the smoothing coefficient.

IV.2. *Stability*

Only the special case of one group of buyers will be considered here; thus $k=1$ and the subscript i will be deleted. Moreover, when talking about stability in this context, we mean *asymptotic stability*. Intuitively speaking, a stationary state is called asymptotically stable if, when given a small perturbation in the state space away from the stationary state, it asymptotically (in time) returns to the stationary state. Thus asymptotic stability is a local characteristic, telling only how the system behaves in a small neighborhood around the stationary state. More exactly, we use the (standard) definition of asymptotic stability as it is given in e.g. Hale (1969).

In the preceding subsection it was shown that if $\lambda<\phi$, then $x_1^*(\sigma)$ approaches a positive value as the smoothing coefficient σ approaches zero, while $x_1^*(\sigma)$ approaches zero for $\lambda\geqslant\phi$. This motivates a division of the stability analysis into two cases. For the case $\lambda<\phi$, a sufficient condition for stability is that the queuing propensity function f is "smooth" for all positive queuing times. For the opposite case, $\lambda\geqslant\phi$, it is sufficient that f is "flat" at zero queuing time.

Proposition 2. Consider a buyer population consisting of only one group, $k=1$, and assume $a(\pi)>0$.

(*a*) Suppose $\lambda < \phi$. Assume that A1–A5 hold and moreover that the queuing propensity function f has a second derivative f'' for every $w > 0$. There then exists an $\varepsilon > 0$ such that the stationary state is asymptotically stable for every smoothing coefficient $\sigma \in (0, \varepsilon)$.

(*b*) Suppose $\lambda \geqslant \phi$. Assume that A1–A5 hold and moreover that the queuing propensity function f is identically equal to unity in some interval $(0, \delta)$. Then the stationary state is asymptotically stable for every smoothing coefficient $\sigma \in (0, \lambda \cdot \delta)$.

As already mentioned, the above proposition does not tell us how the system reacts to large perturbations away from its stationary state. So far, we have no general results concerning the global behavior of the system. However, for the special case where the possibility of postponement is excluded, it may be shown that the stationary state is indeed globally stable, i.e. the system returns to its stationary state at arbitrarily large perturbations.

Proposition 3: Consider a buyer population consisting of a single group without possibility of postponement, i.e. $k=1$, $a(\pi)>0$, $c(\pi)=1$ and $x_4(0)=0$. If assumptions A1–A5 are satisfied, then the system converges asymptotically to its stationary state from any initial state.

As a complement to the analytical stability studies above concerning the special case of one buyer group ($k=1$), a few numerical computer simulations have been made for the case of two buyer groups ($k=2$). Admitting that we have made no extensive simulation studies, we are able to say that all simulations so far show global stability of the system. On the basis of our simulations we make the following conjecture.

Conjecture. At least for the case of two buyer groups, $k=2$, there is a fairly wide class of exogenous parameters and functions satisfying assumptions A1–A5, for which the corresponding stationary states are globally stable.

IV.3 *Walrasian and Non-Walrasian Long-term Equilibrium*

When $x_{1i}=x_{1i}^*, \ldots, x_{4i}=x_i^*$, $i=1, \ldots, k$, the system is in its *normal state*. Some words of explanation and interpretation concerning the term "normal" will be in order.

In an empirical descriptive interpreation of the model, the implication is this: the normal value is the *intertemporal mean* of a state variable. Hence our model is suitable only as a description of a stagnating market. Our conjecture is, however, that the results can be generalized for systems where supply, trade and consumption are changing (e.g. increasing) over time. (One might, for example, consider "immigration" of new potential buyers into the market for good G.) In that case we have to redefine the concept of the normal state, which becomes a relative term ($x_{ji}(t)/n_i(t) = c$ for all t and every i, j). In subsequent comments we refer to the generalized interpretation of the term "normal state", where the stationary state of our model is only a special case.

For a meaningful interpretation, two formally distinct problems—existence and stability—are thoroughly interconnected. To call any intertemporal mean a "normal value" would be a tautological re-naming. What makes an intertemporal mean a "normal value" indeed is the operation of a *feedback mechanism*, assuring that the system, departing from its normal state, will be brought "back to normal". In our simple model the signal guiding the feedback mechanism is w, the queuing time. If queuing takes too long a time, buyers will refrain from joining. If, on the contrary, queuing time is shorter than normal, this will attract more people to join the queue.

In addition to existence and stability, we also have a proposition of *uniqueness* of the stationary state. This is not an indispensable implication of the concept of a normal state. Our uniqueness proposition is due, among other assumptions, to the deterministic framework of the model. In a stochastic setting, the (unique) stationary state in the present deterministic model must be replaced by a (unique) stationary probability distribution for the state of the system.

The normal state could also be called the *long-term equilibrium* of the system.[1] There is, however, some terminological confusion and vagueness in the economic literature, because the term "equilibrium" has strong connotations in the traditions of the profession. Many economists are inclined to restrict the use of this term to denominate a system in *Walrasian* equilibrium. Let us illustrate the problem by considering the present model. The market is in a kind of long-term Walrasian equilibrium if $x_1 = 0$ and $\dot{x}_1 = 0$ at all times. Under certain conditions, which we discuss later on, this may be the case. But then there are other, non-Walrasian equilibria as well, including normal states with positive queues. *The Walrasian equilibria here only constitute a subset of the set of normal states.*

Many economists would call such steady states *dis*equilibria. Precisely the new stream of research referred to in the introduction is usually called "disequilibrium theory".[2] This is not merely a semantic question; in the mind (or in the back of the mind) names are usually associated with value judgements. To put it in very simple terms: 90 out of 100 economists will consider equilibrium as "good", worthwhile to be maintained, or if lost, to be restored. Therefore, "disequilibrium" is something "bad", and therefore it must be avoided. And if "disequilibrium" becomes long-lasting and chronic, it is a sign of degeneration, a perverse state of the system; it is something "abnormal".

We like the term "normal value" as a synonym for "long-term equilibrium

[1] Malinvaud advocates the same approach to the equilibrium concept in his book (1976).
[2] See Barro & Grossman (1971), Benassy (1975) and others.

value", or "steady-state value", because it indicates a descriptive statement, without judgement. The characteristics of a normal state are *system specific.*

When we say: there are systems which will have queues in their normal state, this implies: there are no feedback mechanisms, no social forces in the system driving it toward a Walrasian state. On the contrary, such an economy has some important properties, deeply embedded in the very nature of the system, which permanently restore the normal length of the queue, and so on.

Any normal state, including non-Walrasian equilibria, can be self-reproducing, self-perpetuating only because it is *accepted as normal* by the members of the system. Queuing, waiting, postponing purchases in spite of available financial means, forced substitution—these are social costs, paid by the buyer, in addition to the regular price, paid in money. The propensities to join the queue, to implement forced substitution, to postpone the purchase, i.e. our functions f_i, c_i, and $(1-c_i)$, respectively, express the extent to which the buyers are willing to pay these non-financial social costs of getting the desired goods. They indicate the socially institutionalized acceptance of the conditions prevailing on the market.

V. Dependency of the Normal State on Exogenous Data

V.1. *Introductory Remarks*

We now turn to a study of the normal state in the limiting case $\sigma \downarrow 0$, the state variables thus being given in corollaries 1.1 and 1.2. We will compare normal state stocks and flows corresponding to different collections of parameter values and functions satisfying assumptions A1–A5. Although we are dealing with a dynamic model, a comparison of different normal states of the system leads to results similar to the usual *comparative static* exercises.

First of all the crucial parametric quantity ϕ defined in equation (4.1) requires a closer look. According to corollaries 1.1 and 1.2 this quantity is the *minimal queue clearing service capacity*, i.e. if the service capacity λ is less than this number, then there will be a queue in the normal state, while there will be no queue in the normal state if λ equals or exceeds ϕ. Observe that ϕ depends only on the relative price π, the initial buying propensity functions a_i, the need renewal rates γ_i and $\varkappa_i$ and the sizes of the buyer groups n_i, while it is independent of the queuing propensities f_i, forced substitution propensities c_i, reconsideration rates ϱ_i and of course of the service capacity λ. The quantity ϕ thus reflects the buyers' attitude concerning price, and their consumption rates. Because of this role of ϕ it is natural to relate it to the concept of demand, and actually ϕ may be interpreted in terms of long-term potential demand. Namely, for any normal state with no queuing, the inflow (per time unit) of buyers to the shop or service facility is precisely ϕ; cf. Fig. 2 and corollary 1.2. Thus, for any given collection of parameters and functions describing the buyers' attitude and behavior, the quantity ϕ is the number of requests for good G per time unit that would be made by this population of buyers if the system were in a normal state without queuing. Therefore the quantity may be called the *potential demand* (flow), generated by the buyer population, where each buyer requests one unit of the good at each purchase. (The quantity ϕ generally differs from the flow of potential requests in a particular normal state with queuing. The latter flow, consisting of buyers that would request good G if they did not have to queue, may be calculated from the equations in corollary 1.1.)

Having considered the meaning of the quantity ϕ, we now return to a study of the dependency of the normal state on the service capacity, the relative price and some components of the buyers' attitude. In such a study there are many aspects of the normal state that require consideration. A natural description of the normal state is simply the distribution of the buyers over the four possible states "queuing", "G-satisfied", "H-satisfied" and "postponing", as specified by the normal values of the state variables themselves. However, as a complement to these quantities one may also consider the flow of potential customers, i.e., the flow of buyers that would buy good G if it were available without queuing (we think of the flow into the last decision point before the queue in Figs 1 and 2). In general, this flow splits into three sub-flows: one going to the queue, the other to forced substitution and the third to postponement. In a normal state, these flows represent the shares ($i=1, 2, ..., k$)

$$(fq)_i^* = f_i(w^*) \quad \text{(to queuing)} \tag{5.1}$$

$$(fs)_i^* = (1-f_i(w^*))\cdot c_i(\pi) \quad \text{(to forced substitution)} \tag{5.2}$$

$$(fp)_i^* = (1-f_i(w^*))\cdot(1-c_i(\pi)) \quad \text{(to postponement)} \tag{5.3}$$

of the flow of potential customers of type i. This division of the flow may be seen as the buyers' choice of non-financial costs in face of shortage: whether to spend time queuing, buy a less preferred good or refrain from purchase. In the "non-shortage case" $\lambda \geqslant \phi$, we have $w^*=0$ and thus $(fq)_i^*=1$, $(fs)_i^*=(fp)_i^*=0$ for all i. In the "shortage case" $\lambda<\phi$, however, $w^*>0$ and all shares may be positive. As to forced substitution, it may be of interest not only to know the flow shares $(fs)^*$ ($i=1, 2, ..., k$) but also the stock shares, i.e. the number of forced substitutioners among all substitutioners. For an arbitrary normal state, let this share for buyer group i be denoted r_i^* ($i=1, 2, ..., k$). Through the equations in corollaries 1.1 and 1.2 we get the following expression:

$$r_i^* = \frac{a_i(\pi)\cdot c_i(\pi)\cdot(1-f_i(w^*))}{c_i(\pi)\cdot(1-f_i(w^*))+(1-a_i(\pi))\cdot f_i(w^*)} \quad i=1, 2, \dots, k \tag{5.4}$$

with r_i^* defined as zero if the numerator is zero. In particular, we see that in the "non-shortage case" $w^*=0$ and thus $r_i^*=0$ for all i.

A detailed study of all the aspects of the normal state would be very lengthy, to say the least. However, as the normal values of all state variables and indicators are more or less directly related to the normal queuing time w^*, we may focus on this fundamental characteristic in the following analysis without too much loss of completeness.

The normal queuing time w^* is determined in corollaries 1.1 and 1.2 through the identity $w^* = x_1^*/\lambda$. For convenience we restate the result here. Let the function G: $R_+ \to R_+$ be defined through

$$G(w) = \sum_{i=1}^{k} \frac{a_i(\pi) \cdot n_i}{A_i(w) + a_i(\pi) \cdot w} \tag{5.5}$$

Corollary 1.3. If $\lambda < \phi$, then $w^* > 0$ and w^* is the unique solution to the equation $G(w) = \lambda$. If $\lambda \geqslant \phi$, then $w^* = 0$.

V.2. *Dependency on the Service Capacity*

Next, we study how the normal queuing time w^* depends on the service capacity λ, all other parameters and functions being fixed (in particular thus ϕ being a constant). Let us compare the normal queuing time corresponding to a lower service capacity with the normal queuing time corresponding to a higher service capacity. Intuitively one would expect the queuing time to be shorter in the case with the higher service capacity. It follows directly from the monotonicity of the function G that in fact this is the case.

Observation 1. The normal queuing time w^* is a continuous function of the service capacity λ. For $\lambda \in (0, \phi)$ it is positive and decreasing, while for $\lambda \geqslant \phi$ it is constantly equal to zero.

The normal queue length x_1^* requires a comment. At first glance, one might believe that the qualitative result above would hold also for the normal queue length, i.e. a higher service capacity would yield a shorter queue. However, in the present model we have assumed that it is the queuing time, and not the number of persons in the queue, that influences a potential customer's propensity to join the queue. Therefore, the normal queue length may be related to the service capacity in a non-monotonic way under fairly reasonable assumptions about the queuing propensity functions. This is the case, for example, if there is a finite upper bound on acceptable queuing time, or, more precisely, if there is a finite w_0 such that $f_i(w_0) = 0$ for $i = 1, 2, ..., k$.

Observation 2. The normal queue length x_1^* is a continuous function of the service capacity λ. For $\lambda \in (0, \phi)$ it is positive and for $\lambda \geqslant \phi$ it is constantly equal to zero. If there is a finite upper bound on acceptable queuing time, then $\lim_{\lambda \downarrow 0} x_1^* = 0$.

Thus, as x_1^* is a positive and continuous function of λ for $\lambda \in (0, \phi)$, it cannot be monotonically decreasing throughout the whole interval $(0, \phi)$. The

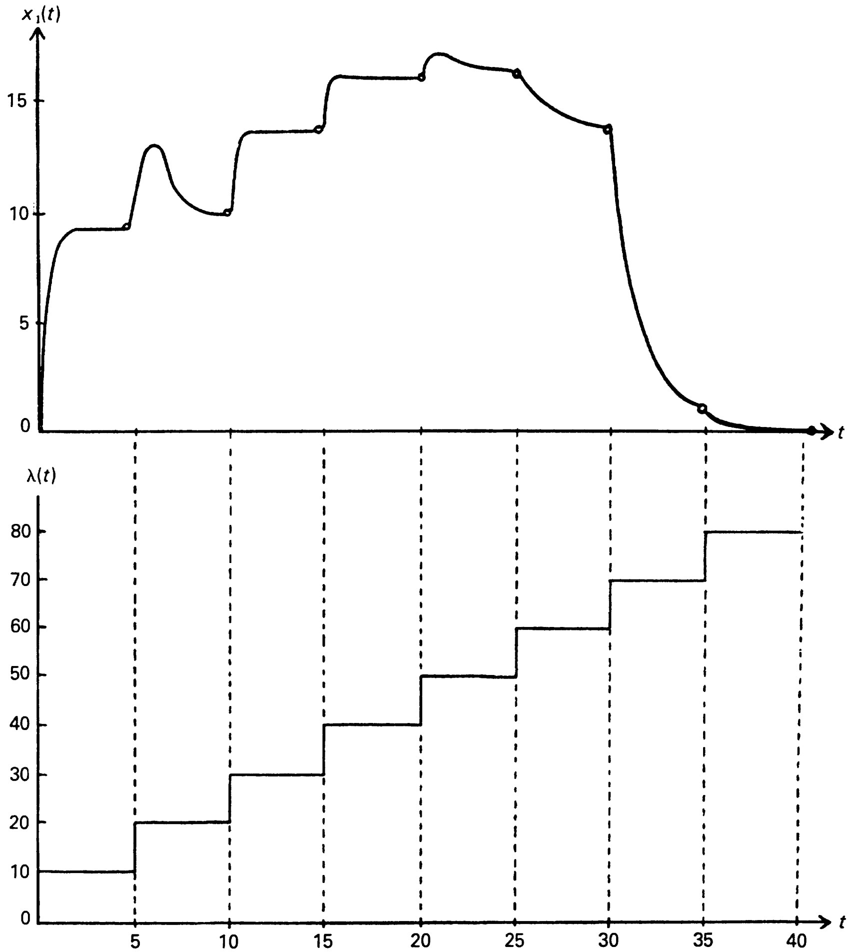

Fig. 3. Typical response in queue length ($x_1(t)$), due to successive, sudden increases in the service capacity ($\lambda(t)$). Here t denotes time and the rings denote the successive normal queue lengths.

dependency of the queue length on the service capacity in a typical case is illustrated in Fig. 3.

In sum: *a higher service capacity yields a shorter normal queuing time but not necessarily a shorter queue.*

V.3. *Dependency on the Price*

We now study how the normal state depends on the relative price π, all other parameters and functions being fixed (in particular λ being a constant). Before studying the normal state variables, however, we should make some observations about the "long-term potential demand" ϕ. From the definition in equation (4.1) it is easily verified that ϕ is a continuous and non-increasing function of π. Moreover, $\phi \to 0$ as $\pi \to +\infty$. Thus, as $\lambda > 0$, $\phi(\pi) \leqslant \lambda$ for some

$\pi \in R_+$. Let π_0 denote the minimal price satisfying this inequality (the existence of a minimal price being guaranteed by the continuity of ϕ). In force of corollaries 1.1 and 1.2 this leads to the following result concerning the existence or non-existence of a queue in the normal state.

Observation 3. For any fixed service capacity and collection of parameters and functions describing the buyers' attitude, there exists a *minimal queue clearing price*, i.e. a finite price π_0 satisfying

$$\pi < \pi_0 \Rightarrow x_1^* > 0$$

$$\pi \geqslant \pi_0 \Rightarrow x_1^* = 0$$

In other words, there is always a relative price high enough to make the corresponding normal state queue-less. However attactive such a normal state may seem, it should be observed that although there is no queuing at prices above π_0, the situation is not better in terms of the number of persons served. It is easily verified from corollaries 1.1 and 1.2 that the normal service rate s^* satisfies the equation $s^* = \min(\lambda, \phi)$. Thus, as a function of the relative price π, the normal service rate is constantly equal to λ in the price interval $(0, \pi_0]$ while it declines with ϕ in the price interval $(\pi_0, +\infty)$.

As the minimal queue clearing price π_0 renders the long-term potential demand ϕ equal to the service capacity λ, it can be regarded as the *Walrasian market clearing price*. This price is unique in the deterministic framework of the present model. Below that price there will always be a queue, above it there will never be a queue, *ceteris paribus*.

Let us now consider what happens to the normal queuing time if a low relative price is raised. Intuitively one would expect the normal queuing time corresponding to a higher price not to exceed the normal queuing time corresponding to a lower price. In terms of the present model, this relationship indeed holds.

Observation 4. The normal queuing time w^* is a continuous function of the relative price π. For prices in the interval $(0, \pi_0)$ it is positive and non-increasing, while it is constantly equal to zero for prices $\pi \geqslant \pi_0$.

In sum: *a higher relative price never yields a longer normal queuing time, and there is always a relative price high enough to make the corresponding normal state queue-less.*

V.4. *Dependency on the Queuing and Forced Substitution Propensities*

In the two preceding subsections we have studied how the normal state depends on such "market control variables" as relative price and service capacity. We now study how the normal state depends on a few components of the buyers' attitude.

Let us first consider the dependency of the normal queuing time on the queuing propensities of the buyers, all other parameters and functions being fixed. Let $f_1, f_2, ..., f_k$ and $g_1, g_2, ..., g_k$ be two alternative collections of queuing propensity functions. If, for every i, $f_i(w) \geqslant g_i(w)$ for all $w > 0$, and, for some i, $f_i(w) > g_i(w)$ for all $w > 0$, then the collection of functions $[f_i]$ is said to *dominate* the collection $[g_i]$.

Observation 5. Suppose $\lambda < \phi$ and that one collection of queuing propensity functions $[f_i]$ dominates another collection of queuing propensity functions $[g_i]$. Then the normal queuing time corresponding to the first collection exceeds the normal queuing time corresponding to the second.

In other words: *higher queuing propensities yield a normal state with a longer queuing time.*[1]

Next, let us consider the dependency of the normal queuing time on the forced substitution propensities of the buyers, all other parameters and functions being fixed (in particular the relative price being fixed). Let $[c_i]$ and $[d_i]$ be two alternative collections of forced substitution propensity functions, and suppose $[c_i]$ majorizes $[d_i]$, i.e. for every i $c_i(\pi) \geqslant d_i(\pi)$ for all $\pi \geqslant 0$.

Observation 6. Suppose that one collection of forced substitution propensity functions $[c_i]$ majorizes another collection of such functions $[d_i]$. At any fixed relative price π, the normal queuing time corresponding to $[c_i]$ is less than or equal to the normal queuing time corresponding to $[d_i]$.

In other words: *higher forced substitution propensities never yield a longer normal queuing time.*

Observations 5 and 6 support the remark made at the end of Section IV.3. The constellation of the state variables will depend on the attitude of the buyers in the different groups. Also, *there is some "trade-off" between the different non-financial costs of shortage.* As such costs we have in mind queuing, postponement and forced substitution. Observation 6 illustrates one such trade-off. The buyers can achieve a shorter queuing time if they are more willing to accept forced substitution. In general, a decrease in one of these costs, without an increase in other non-financial costs, can be assured only by changing the ultimate determining factors: the consumption and voluntary substitution patterns on one hand and/or the service capacity and price on the other.

As a final remark we note that shifts in the "market control variables" λ and π, as well as shifts in the buyers' attitude, in general have distributional

[1] Perhaps this sounds self-evident. However, we want to remind the reader of the direction of causality. Queuing propensity can be a decision variable of the consumer, but queuing time will be the joint consequence of the individual decisions, and as such, it will be given to each individual. The actual shortage situation depends on the tolerance of the buyers.

effects across buyer groups. For instance, an increase in the relative price may redirect more price-sensitive buyer groups to substitution, while other, less price-sensitive groups may only get a shorter queuing time without changed consumption pattern. Also, an increased propensity for forced substitution in one buyer group may benefit other groups through a shorter queuing time.

Thus, in addition to the problem of distribution of income, in money terms, which is well studied in the literature, we have here another important aspect: the distribution of non-financial social costs of consumption among the different groups of the population.

VI. Further Extensions

This paper is only a first step in the study of a large set of problems. The analysis is kept within very narrow boundaries, using drastic simplifications, mainly for expository purposes. Extentions and variations are needed. Here are some of the possible further directions of research:

— A market with an increasing volume of trade over time.
— Endogenous determination of supply, including inventory stocks.
— Stochastic interaction between the agents.
— More complex organizational structure of the market, e.g. many sellers instead of one.
— Alternative activities; e.g. the buyers searching for goods at different selling places or at different points in time.
— The buyers' information about the supply and the sellers' information about demand, either exogenously provided or endogenously generated.
— Queue priorities among buyer groups.
— Purchase quantity depending on the price and/or the queuing or search time.
— Recycling of durable components of consumed items.

A forthcoming paper will discuss the case of search, partly including a few of the extensions listed above.

Appendix

For editorial reasons, only fragmentary proof sketches for some of the mathematical propositions can be given here. Full proofs are presented in a mimeographed version of the paper. The mimeographed paper can be obtained from the authors or the Institute for International Economic Studies, Stockholm University, on request.

Proposition 1. Stationarity implies the equation $F_\sigma(x_1)=1$, where

$$F_\sigma(x)=\sum_{i=1}^{k}\frac{a_i\cdot n_i}{A_i(x/\lambda)\cdot\lambda\cdot h_\sigma(x)+a_i\cdot x}.$$

By A1–A5, F_σ is continuous and (strictly) decreasing with $F_\sigma(0)=+\infty$ and $F_\sigma(n)<1$, and thus the equation $F_\sigma(x_1)=1$ has a unique root $x_1^*(\sigma)$ in the interval $(0, n)$.

Proposition 2. The system of differential equations is linearized at the stationary state; cf. Chapter II, corollary 6.1 in Hale (1969).

Proposition 3. The result may be obtained by a geometric analysis of the (x_1, x_2)-phase plane.

Observation 2. For $\lambda<\varphi$, x^* is the root to the equation $H_\lambda(x)=1$, where $H_\lambda(x_1)$ denotes the left-hand side in (4.3). From the hypothesis of a finite upper bound w_0, it follows that $H_\lambda(x)=0$ for every $x\geqslant\lambda\cdot w_0$. As H_λ is continuous with $H_\lambda(0)=\varphi/\lambda>1$, we have $x_1^*<\lambda\cdot w_0$ for every λ and thus $x_1^*\to 0$ as $\lambda\to 0$.

Observation 4. A sufficient condition for the monotonicity result is that the quantity $G(w)$, for any fixed $w>0$, does not increase with π. A study of the price dependency of $A_i(w)/\alpha_i(\pi)$ gives the condition.

References

Barro, R. J. & Grossman, H. I.: A general disequilibrium model of income and employment. *American Economic Review 61*, 82–93, 1971.

Barro, R. J. & Grossman, H. I.: Supressed inflation and the supply multiplier. *Review of Economic Studies 41*, 87–104, 1974.

Benassy, J. P.: Neo-Keynesian disequilibrium theory in a monetary economy. *Review of Economic Studies 42*, 503–523, 1975.

Clower, R. W.: The Keynesian counterrevolution: A theoretical appraisal. In F. H. Hahn and F. P. R. Brechling (eds.), *The theory of interest rates*, Proceedings of an IEA Conference. Macmillan, London, 1965.

Hale, J.: *Ordinary differential equations.* Wiley, New York, 1969.

Kleinrock, L.: *Queuing systems*, vol. II. Wiley, New York, 1976.

Kornai, J.: *Anti-equilibrium.* North-Holland, Amsterdam, 1971.

Kornai, J.: *Az adaptáció csikorgó gépezete* (The grating machinery of adaptation), in Hungarian, mimeographed. Institute of Economics, Budapest, 1974.

Kornai, J.: On the measurement of shortage. *Acta Oeconomica* (forthcoming 1977).

Malinvaud, E.: *The theory of unemployment reconsidered*, Basil Blackwell, Oxford, 1976.

McCafferty, S.: "Excess demand, search and price dynamics". *American Economic Review 67* (2), 228–235, 1977.